CHAPTER 4

4.1. The sketches are as below

![Sketches showing frequency responses](image)

Fig. S4-1.

4.2. It can be verified that the decimator and interpolator can be interchanged if M and L are relatively prime. Thus the system becomes as shown below,

![System diagram](image)

Fig. S4-2.

and therefore,

\[ y(n) = \begin{cases} 
  x(n) & n \text{ even} \\
  0 & n \text{ odd} 
\end{cases} \]

or, \( y(n) = 0.5[1 + (-1)^n]x(n) \).

4.3. We have to first squeeze the frequency response and then filter out the unwanted interpolation images. The scheme to recover \( x(n) \) from \( y(n) \) is

![Scheme diagram](image)

Fig. S4-3.
4.7. We know that \( W^k, 0 \leq k \leq M - 1 \) is a set of \( M \) distinct numbers. Since \( W^M = 1 \), the set
\[
W^k, 0 \leq k \leq M - 1,
\] 
(S4.7a)
is evidently a subset of the set \( W^k, 0 \leq k \leq M - 1 \). So these sets are identical if and only if \( W^{k_1} \neq W^{k_2} \) for \( k_1, k_2 \) such that \( 0 \leq k_1 < k_2 \leq M - 1 \), i.e.,
\[
e^{-j2\pi k_1 L/M} \neq e^{-j2\pi k_2 L/M}, 0 \leq k_1 < k_2 \leq M - 1,
\] 
(S4.7b)
i.e., if and only if \((k_2 - k_1)L \neq KM\) for integer \( K \) i.e., if and only if
\[
\frac{L}{M} \neq \frac{K}{k_2 - k_1}.
\]
If \( L \) and \( M \) are relatively prime, this is not possible (since \( 0 < k_2 - k_1 < M \)). On the other hand, if \( L \) and \( M \) are not relatively prime, we can write
\[
\frac{L}{M} = \frac{K}{k_2}
\]
for appropriate \( K \) and \( k_2 < M \), so that \( W^{k_1} = W^{k_2} \) and (S4.7b) fails for \( k_1 = 0 \). Summarizing, the two given sets are identical if and only if \( L \) and \( M \) are relatively prime.

4.8. We get \( y_2(n) = x(nM/L) \) if \( nM \) is a multiple of \( L \) and 0 otherwise. Hence for \( y_1(n) = y_2(n) \) to hold, \( nM \) should be a multiple of \( L \) if and only if \( n \) is a multiple of \( L \). This means that \( M \) and \( L \) should be relatively prime.

4.9. Coins painted are numbered
\[
kM, \quad k = 0, 1, \ldots
\]
The painter returns to the 0th coin without having completed all the coins if and only if
\[
kM \mod N = 0
\]
for some \( k < N \). This holds if and only if \( kM = nN \) for some integer \( n \), so that
\[
\frac{M}{N} = \frac{n}{k}.
\]
Since \( k < N \), this is equivalent to saying that there is a common factor > 1 between \( M \) and \( N \), i.e., \( M \) and \( N \) are not relatively prime. The conclusion is that the painter covers all coins before returning to coin 0, if and only if \( M \) and \( N \) are relatively prime.

4.10. \( x(n) \) has period \( N \) and \( y(n) = x(Mn) \). Let \( L \) denote the period of \( y(n) \). Now,
\[
y(n) = y(n + L) \quad \forall n
\]
\[
\iff x(Mn) = x(Mn + ML) \quad \forall n
\]
\[
\iff ML \text{ is a multiple of } N
\]
\[
\iff pmL \text{ is a multiple of } pq \quad \text{(let } \gcd(M, N) = p, M = pm, N = pq\text{)}
\]
\[
\iff mL \text{ is a multiple of } q
\]
\[
\iff L \text{ is a multiple of } q \quad \text{(since } m \text{ and } q \text{ are relatively prime)}
\]

SOLUTIONS TO CHAP. 4 PROBLEMS 3
L should be the smallest integer such that L is a multiple of q. Therefore,

\[ L = q = \frac{N}{p} = \frac{N}{\gcd(M, N)} = \frac{\lcm(M, N)}{M}. \]

Obviously, when M and N are relatively prime, L = N is the largest possible value.

4.11. The plots are

![Fig. S4-11.](image)

4.12.

a) \( h(n) = (1/2)^n \) for \( 0 \leq n \leq 9 \). So

\[ E_0(z) = h(0) + h(2)z^{-1} + h(4)z^{-2} + h(6)z^{-3} + h(8)z^{-4}, \]

\[ = 1 + (1/2)^2 z^{-1} + (1/2)^4 z^{-2} + (1/2)^6 z^{-3} + (1/2)^8 z^{-4}, \]

\[ E_1(z) = h(1) + h(3)z^{-1} + h(5)z^{-2} + h(7)z^{-3} + h(9)z^{-4}, \]

\[ = (1/2) + (1/2)^3 z^{-1} + (1/2)^5 z^{-2} + (1/2)^7 z^{-3} + (1/2)^9 z^{-4}. \]

b) For \( g(n) = \alpha^n u(n) \), its transfer function can be written as

\[ G(z) = \frac{1}{1 - \alpha z^{-1}} = \frac{1}{1 - \alpha^2 z^{-2}} + \frac{\alpha z^{-1}}{1 - \alpha^2 z^{-2}}. \]

By using this, we can obtain the polyphase components of \( H(z) \) as follows,

\[ E_0(z) = \frac{1}{1 - (1/4)z^{-1}} + \frac{(1/3)^4 z^{-2}}{1 - (1/9)z^{-1}}, \]

\[ E_1(z) = \frac{1/2}{1 - (1/4)z^{-1}} + \frac{(1/3)^3 z^{-1}}{1 - (1/9)z^{-1}}. \]

4.13. Because \( 1/(1 + \alpha z^{-1}) = (1 - \alpha z^{-1})/(1 - \alpha^2 z^{-2}) \), \( H(z) \) can be written as \( E_0(z^2) + z^{-1} E_1(z^2) \) with the polyphase components

\[ E_0(z) = \frac{a(1 - z^{-1})}{1 - a^2 z^{-2}}, \quad E_1(z) = \frac{1 - a^2}{1 - a^2 z^{-1}}. \]

Note that neither polyphase component is allpass (even if \( a \) is real so that \( H(z) \) is allpass).

4.14. We can rewrite

\[ H(z) = \frac{1 + 2R \cos \theta z^{-1} + R^2 z^{-2}}{(1 - R^2 z^{-2})^2 - 4R^2 \cos^2 \theta z^{-2}}. \]