Robust Adaptive Beamforming with application to Matched Field Processing

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Outline

- **Background**
- **4 Robust Adaptive Beamforming Methods**
  - LCMP
  - White Noise Gain Constraint
  - Gershman *et al.*
  - Stoica *et al.*
  - Simulations
- **Intro to Matched Field Processing**
  - MFP simulations
  - MFP results on the actual data
- **Conclusion**
To Start with...

Shortcomings:

- Does not provide sufficient robustness against mismatch between presumed and actual signal steering vector.
- Tends to suppress the SOI by adaptive nulling.
Effect of mismatch in beamscan

No mismatch case

Mismatch case

15-elt ULA
NL = 50 dB
SNRin = 63 dB

Source
SL = 140 dB
R = 7 km
Steering Vector Mismatch Due to…

- look direction mismatch
- array perturbation
- array manifold mismodeling
- wavefront distortions
- source local scattering
- …
Approaches to Robust Adaptive Beamforming

- **Linear** Constraint Minimum Power Beamformer

  \[ C^H w = g \]

- Directional Constraints
White Noise Gain Constraint

\[ R = \sigma_s^2 v_s v_s^H + \sigma_n^2 S_n \]

- **Array Gain:**
  \[ G = \frac{\|w^H v\|^2}{w^H S_n w} \quad \text{distortionless, white noise} \quad \frac{1}{\|w\|^2} \]

- **Signal perturbed by uncorrelated random errors:**
  \[ R = \sigma_s^2 \left( v_s v_s^H + \xi I \right) + \sigma_n^2 S_n \]

- **Sensitivity:**
  \[ S = \left( \frac{dG}{d\xi} \right) \quad \text{distortionless} \quad \|w\|^2 = \frac{1}{G} \]

- Sensitivity increases as white noise gain decreases
- **UWLA:** low sensitivity; **MVDR:** high sensitivity
White Noise Gain Constraint

- **Goal**: reduce sensitivity

- Impose **quadratic** constraint on white noise gain to increase robustness

  \[
  \min_w \ w^H R w \quad \text{subject to} \quad w^H d = 1, \quad \text{and} \quad \|w\|_{\infty}^{-1} \leq \delta^2.
  \]

- **Solution is**

  \[
  w_{WNGC} = \frac{(R + \varepsilon I)^{-1} d}{d^H (R + \varepsilon I)^{-1} d}
  \]

  No simple relation

- Adjust \( \varepsilon \) until white noise gain constraint is satisfied
Shortcoming of WNGC

- Relationship between $\varepsilon$ and $\delta^2$ is not simple
- Need approach to compute the $\varepsilon$ based on the uncertainty of the steering vector
- Iterative procedure is required to adjust the diagonal loading factor
- In practice, can use *ad hoc* method to determine it
Looking for methods that…

- Have sufficient robustness against arbitrary steering vector mismatch
- Have sound mathematical framework
- Are computationally easy to implement

One approach:

2003: Sergiy A. Vorobyov, Alex B. Gershman, and Zhi-Quan Luo
Steering vector distortion: $\Delta$

$$\|\Delta\| \leq \varepsilon$$

The actual steering vector belongs to

$$A(\varepsilon) \triangleq \{ \mathbf{c} \mid \mathbf{c} = \mathbf{v} + \mathbf{e}, \|\mathbf{e}\| \leq \varepsilon \}$$
Worst-Case optimization

For all vector in $A(\varepsilon)$, the array response should NOT be smaller than 1

$$|w^Hc| \geq 1 \quad \text{for all } c \in A(\varepsilon)$$

$A(\varepsilon) \overset{\Delta}{=} \{ c | c = v + e, \|e\| \leq \varepsilon \}$
Formulation of Robust BF

\[ \min_w w^H R w \ \text{subject to} \ \left| w^H c \right| \geq 1 \ \text{for all} \ c \in A(\varepsilon) \]

- Looks good, but
  - Non-linear
  - Non-convex
  - Infinite # of constraints

\[ \min_w w^H R w \ \text{subject to} \ \left| w^H \left( I + \lambda \varepsilon^2 I \right) v \right| \geq 1 \]

\[ \min_w w^H R w \ \text{subject to} \ w^H v \geq \varepsilon \| w \| + 1 \ \& \ \text{Im} \{ w^H v \} = 0 \]
Another Method: Stoica et al. 2003: Directly Estimate Signal Power $\sigma_s^2$

$$R = \sigma_s^2 v_s v_s^H + \sum \sigma_i^2 v_i v_i^H + S_n$$

- Using $w_0^H R w_0$ as $\sigma_s^2$ estimate, MPDR gives

$$\sigma_s^2 \sim \frac{1}{v_s^H R v_s}$$

- It can be shown that it’s the solution to

$$\max_{\sigma^2} \sigma^2 \quad \text{subject to} \quad R - \sigma^2 v_s v_s^H \geq 0$$

Covariance fitting problem
Adding in uncertainty

\[
\max_{\sigma^2} \sigma^2 \quad \text{subject to} \quad R - \sigma^2 v_s v_s^H \geq 0
\]

- Given an uncertainty “ellipsoid”
  \[
  (v - v_p)^H C^{-1} (v - v_p) \leq 1
  \]

- Estimate the power of SOI by

\[
\max_{\sigma^2, v} \sigma^2 \quad \text{subject to} \quad R - \sigma^2 vv^H \geq 0
\]

\[
& (v - v_p)^H C^{-1} (v - v_p) \leq 1
\]
By derivation, we get

\[
\max_{\sigma^2, \nu} \sigma^2 \quad \text{subject to} \quad R - \sigma^2 \nu \nu^H \succeq 0
\]

\[
& (\nu - \nu_p)^H C^{-1} (\nu - \nu_p) \leq 1
\]

\[
\min_{\nu} \nu R^{-1} \nu^H \quad \text{subject to} \quad (\nu - \nu_p)^H C^{-1} (\nu - \nu_p) \leq 1
\]

Assume \( C = \varepsilon I \), we have

\[
\min_{\nu} \nu R^{-1} \nu^H \quad \text{subject to} \quad \|\nu - \nu_p\|^2 \leq \varepsilon
\]

**Solution:**

\[
\nu_0 = \left( \frac{R^{-1}}{\lambda} + I \right)^{-1} \nu_p
\]

\[
\omega_0 = \frac{R^{-1} \nu_0}{\nu_0^H R^{-1} \nu_0}
\]

**Direct estimation** of the actual steering vector
Relationship
Between Stoica’s and Gershman’s method

It can be shown that,

\[
\min_v v R^{-1} v^H \quad \text{subject to} \quad \|v - v_p\|^2 \leq \varepsilon
\]

Let \(v_0\) denote the optimal solution, and

\[
w_0 = \frac{R^{-1}v_0}{v_0^H R^{-1}v_0}
\]

Then \(w_0\) is the optimal solution to

\[
\min_w w^H R w \quad \text{subject to} \quad w^H v \geq \sqrt{\varepsilon} \|w\| + 1 \quad \& \quad \text{Im}\{w^H v\} = 0
\]
Simulation 1: Beampatterns of LCMP, WNGC, Stoica and Gershman

\[ \varepsilon_{\text{Stoica}} = 0.3, \quad \varepsilon_{\text{Gershman}} = \sqrt{0.3} \]

Beampattern \( B(\theta) \) in dB

\[ \theta \text{-space} \]

\[ \theta \text{-space} \]
Compare the 4 methods using sample covariance matrix

Beampattern $B(\theta)$ in dB

\begin{align*}
\text{MPDR} & \quad \text{LCMP} \\
\text{MPDR} & \quad \text{WNGC} \\
\text{MPDR} & \quad \text{Stoica} \\
\text{MPDR} & \quad \text{Gershman}
\end{align*}
Comparison of algorithms in beamscan space in presence of mismatch
MFP overview

\[ B(\mathbf{a}) = \mathbf{w}^H(\mathbf{a}) \mathbf{S}(\mathbf{a}_{true}) \mathbf{w}(\mathbf{a}) \]

Peak of the output of the beamformer \( B(\mathbf{a}) \) is at \( \mathbf{a}_{true} \)

\( \mathbf{w} \) depends on the beamformer used
SwellEx’ 96 experiment

Sound speed profile

- Density = 1.76 g/cm³
- Attenuation = 0.2 dB/k.mHz
- $c_{\text{sea}} = 1572.3$ m/s
- $c_{\text{atmos}} = 1593.0$ m/s

4 knots

Source

60 m

21-elt
94.125 to 212.25 m
$\text{d} = 5.63$ m

VLA

23.5 m
Processing of received array data

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<th>Vert. Array</th>
<th>Received Time series</th>
<th>chunk’s FFTs</th>
<th>snapshots</th>
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<td>L</td>
<td><img src="image4" alt="Graph 4" /></td>
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<td>(\ldots)</td>
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chunk k
Spectrogram of data

\[ F_i = 49, 64, 79, 94, 112, 130, 148, 166, 201, 235, 283, 338, 388 \text{ Hz} \]

CSDM estimate,
\[
\hat{S}_i = \frac{1}{K} \sum_{k=1}^{K} X_{ki} X_{ki}^H
\]
Simulation results at $F = 148$ Hz

- $\text{SNR}_{\text{in}} = 10$ dB, 21 element array
- Source at $r = 3000$ m, $z = 60$ m
Simulation results at $F = 148$ Hz

$\text{WNGC} = 0.5N$
Simulation results at $F = 148 \text{ Hz}$
Experimental Results at $F = 49$ Hz

Source at $r = 3000$ m and $z = 60$ m
Experimental Results at $F = 49 \text{ Hz}$

$WNGC = 0.5N$
Experimental Results at $F = 49$ Hz
Results averaged over first 5 frequencies
Results averaged over first 5 frequencies

LCMP Beamformer averaged over first 5 frequencies

WNGC Beamformer averaged over first 5 frequencies

WNGC = 0.5N
Results averaged over first 5 frequencies
Results averaged over all 13 frequencies
Results averaged over all 13 frequencies

LCMP Beamformer averaged over 13 frequencies

WNGC = 0.5N

WNGC Beamformer averaged over 13 frequencies
Results averaged over all 13 frequencies
Conclusions

- Investigated various robust ABF algorithms
- MFP results improved as we average over frequencies (except for MPDR)
- MUSIC best localized the source for this particular set of MFP data
References

- SwellEx-96 Experiment data (http://www.mpl.ucsd.edu/swellex96/).
Normal mode representation of pressure field

The normal mode representation of the field $p(r,z)$ at a range $r$ and depth $z$ from the source is given by

$$
p(r,z) = \frac{ie^{-i\pi^4}}{\rho(z_s)\sqrt{8\pi}} \sum_n U_n(z_s)U_n(z)e^{ik_nr}
$$

where $\rho(z_s)$ is the density at the source depth $z_s$, $k_n$ is the mode propagation constant for mode $n$, and $U_n$ are normalized eigenvectors of the following eigenvalue problem,

$$
\frac{d^2U_n}{dz^2} + \left(K^2(z) - k_n^2\right)U_n(z) = 0
$$

The eigenvectors $U_n$ are zero at $z = 0$, and satisfy the local boundary conditions at the ocean bottom.