Capacity, Diversity and Multiplexing Gain
for MIMO channels

Yuzhe Jin
June 12, 2007
Outline

• Background

• Capacity of MIMO channels
  – Deterministic
  – Fading
  – Outage

• Diversity and Multiplexing Gains
  – Motivation
  – Fundamental tradeoff
  – Two practical schemes

• Extension to Multiple-Access Channels

• Conclusion
Background: SISO Channel

Before we start, let’s recall the discrete-time AWGN channel:

\[ Z \sim \mathcal{N}(0, N) \]

\[ X \rightarrow + \rightarrow Y \]

Figure 1: AWGN Channel

Some important observations:

- A power constraint \( P \) on input \( X \)

- Capacity

\[ C = \max_{p(x), P} I(X; Y) = \log \left( 1 + \frac{P}{N} \right) \]
Background: Multiple Antennas

\[ Z \sim \mathcal{N}(0, N) \]

![Figure 2: Single-input single-output](image)

What if we have multiple antennas at each side?

![Figure 3: Multiple-input multiple-output](image)
Adding in Gaussian noise, we have the channel model:

\[ Y = HX + N \]

where \( X_{m \times 1} \) is the channel input, \( Y_{n \times 1} \) is channel output. \( H_{n \times m} \) is the coefficient associated with each path between a transmit antenna and a receiver antenna, \( N_{n \times 1} \) is zero-mean complex Gaussian noise, and \( E(NN^H) = I \).
Capacity: When $H$ is fixed and known

The channel is $Y = HX + N$. Suppose $E(XX^H) = Q$.

\[
I(X;Y) = H(Y) - H(Y|X) \\
= H(Y) - H(N) \\
= \log \det(I + HQH^H) \\
= \log \det(I + QH^H H) \quad \text{(Property of det)} \\
= \log \det(I + \Lambda^{1/2} U Q U^H \Lambda^{1/2}) \quad \text{(eigen-decomposition)} \\
= \log \det(I + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2}) \\
\leq \prod_i (1 + \tilde{Q}_{ii} \lambda_i)
\]

where $\tilde{Q} = U Q U^H$, and $\Lambda = \text{diag}(\lambda_1, ..., \lambda_m)$. 
Capacity: When $H$ is fixed and known, Cont’d

$$I(X;Y) \leq \prod_i (1 + \tilde{Q}_{ii} \lambda_i)$$

With equality, when $\tilde{Q}$ is diagonal, and optimal diagonal entries can be found via water-filling:

$$\tilde{Q}_{ii} = (\mu - \lambda_i^{-1})^+, \quad i = 1, 2, 3..., m$$

where $\mu$ is chosen to satisfy $\sum_i \tilde{Q}_{ii} = P$. 
Capacity: When $H$ is random

• Suppose each entry of $H$ is independently drawn from a zero-mean complex Gaussian distribution, with independent real and imaginary parts, each with variance $1/2$.

• Gaussian Channel with Rayleigh Fading

• What is the capacity?
Capacity: When $H$ is random

Assumption: $H$ is known at receiver, but not known at transmitter.

$$I(X; (Y, H)) = E_H[\log \det(I + HQH^H)]$$

It’s shown that the optimal input should be a circular symmetric complex Gaussian zero-mean and covariance $(P/m)I$. Then,

$$I(X; (Y, H)) = E_H[\log \det(I + \frac{P}{m}HH^H)]$$

Let $W = HH^H, m \leq n; H^HH, m > n$. $W$ is Wishart Distributed, we can represent the mutual information in terms of the eigenvalues of $W$.

$$I(X; (Y, H)) = \min(m, n) \cdot E[\log(1 + \frac{P}{m} \lambda_1)]$$

Observation: MIMO can be interpreted as multiple parallel spatial channels.
Capacity: When $H$ is random, but realized only once

- Ergodic capacity doesn’t apply to this case. There will always be a positive probability that the realization of $H$ is too bad to support a target rate $R$.

- Definition of Outage
  Outage is defined as the event that the mutual information of the channel does not support a target rate, due to a “bad” realization of channel coefficients $H$.

- Outage Probability: The probability that an outage event occurs.

$$P\left(\underbrace{R}_{\text{desired rate}} > \underbrace{I(X,(Y,H))}_{\text{instantaneous mutual information}}\right) = \epsilon$$

- Question:
  - Given a target rate, what’s the outage probability?
  - Given an outage probability, what’s the max rate we can achieve?
Capacity: For more complicated cases

- When receiver doesn’t know the channel, what is the capacity and corresponding input? and other complicated channel models?

- For more complicated channels, we may not be able to figure out channel capacity in closed form. Meanwhile, the capacity-achieving input distribution is as well hard to characterize.

- Is there other metric that captures the characteristic of a channel, other than capacity?

- In practice, given a coding scheme, is there other metric that can give us more insight about the performance?

- We may need more than capacity result.
In a MIMO Channel with $m$ transmit antennas and $n$ receive antennas,

- Each pair of transmit-receive antennas provides a signal path from the transmitter to receiver.
- If we send signals carrying the same information through different paths, multiple independently faded replicas of the data symbol can be obtained at the receiver.
- More reliable reception can be achieved. ("reliable"?)
Diversity: Reliable Reception

Suppose we have a slow Rayleigh fading channel with a single transmitter and \( n \) receivers,

- \( n \) different fading paths

- The average error probability can be made to decay like \( \frac{1}{\text{SNR}^m} \), at high SNR. Note for a single-antenna fading channel, it’s \( \frac{1}{\text{SNR}} \)

- Use multiple paths to *combat* fading.

- A question: Can fading be beneficial?
Multiplexing

- If the path gains between individual transmit-receive antenna pairs fade independently, the channel matrix will be well-conditioned with high probability.

- Higher rate. (MIMO as multiple parallel spatial channels)

- It's shown that at high SNR regime, the channel capacity is about

\[ C(SNR) = \min(m, n) \log\text{SNR} + O(1) \]

- The channel capacity is increased by using multiple antennas due to spacial multiplexing. We can transmit independent information streams in parallel through the spacial channels.
Can we get both?

Figure 5: A conflict?

Obviously, the conflict between the two suggests a fundamental tradeoff between benefits obtained from diversity and multiplexing.
Consider a family of codes $C'(\text{SNR})$ of block length $l$, one at each SNR level. Let $R(\text{SNR})$ be the rate of the code $C'(\text{SNR})$.

- $C'(\text{SNR})$ is said to achieve diversity gain $d$ if the average error probability
  \[ \lim_{\text{SNR} \to \infty} \frac{P_e(\text{SNR})}{\log \text{SNR}} = -d \]

- $C'(\text{SNR})$ is said to achieve the multiplexing gain $r$ if the data rate
  \[ \lim_{\text{SNR} \to \infty} \frac{R(\text{SNR})}{\log \text{SNR}} = r \]

- For each $r$, define $d^*(r)$ to be the supremum of the diversity advantage achieved over all schemes.
Theorem: Assume $l \geq m + n - 1$. The optimal tradeoff curve $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, \min(m, n)$, where $d^*(k) = (m - k)(n - k)$. In particular, $d^*_{\text{max}} = mn$, $r^*_{\text{max}} = \min(m, n)$.

Figure 6: Diversity-multiplexing tradeoff, for general $m$, $n$, $l \geq m + n - 1$
How do we interpret the Optimal Tradeoff?

- The optimal tradeoff curve intersects the $r$ axis at $\min(m, n)$, i.e. the maximum achievable spatial multiplexing gain $r^*_\text{max}$ is the total number of degrees of freedom provided by the channel as suggested by the ergodic capacity result.

- Intuitively, as $r \to r^*_\text{max}$, the data rate approaches the ergodic capacity.

- At this point, however, no positive diversity gain can be achieved, i.e. no protection against the randomness in the fading channel.
How do we interpret the Optimal Tradeoff?

- The curve intersects the $d$ axis at the maximal diversity gain $d_{\text{max}}^* = mn$, corresponding to the total number of random fading coefficients that a scheme can average over.

- In order to achieve the maximal diversity gain, no positive spatial multiplexing gain can be obtained at the same time.
How do we interpret the Optimal Tradeoff?

- Bridges the gap between the two design criteria

- Positive diversity gain and multiplexing gain can be achieved simultaneously.

- Increasing one comes at a price of decreasing the other.

- This tradeoff curve provides a more complete picture of the achievable performance over multiple-antenna channels than two extreme points corresponding to maximum diversity/multiplexing gain.
How do we interpret the Optimal Tradeoff?

Figure 7: Adding one transmit and one receive antenna increases multiplexing gain by 1 at each diversity level. Curve shifted to the right by 1.
Example: Two Practical Schemes

Repetition scheme vs. Alamouti scheme

\[
X = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}
\]

Figure 8: Diversity-Multiplexing tradeoff and comparison between two schemes.
Beyond the examples

• If the performance of schemes only evaluated by maximal diversity gain $d(0)$, we cannot distinguish the performance of repetition scheme and Alamouti scheme.

• Different coding schemes may be designed for different goals.
  – Orthogonal Designs (e.g. Alamouti): Maximize the diversity gain.
  – V-BLAST: Maximize the spatial multiplexing gain.

• The diversity-multiplexing tradeoff provides a unified framework to make fair comparisons and helps us understand the characteristic of a particular scheme more completely.
Beyond the examples

Figure 9: Orthogonal Designs vs. V-BLAST
Extension to Multiple-Access Channels

• The discussion on diversity-multiplexing tradeoff can be extended to channels with multiple-access.

• Suppose we allow each user to have a diversity gain of $d$, then we can characterize the set of multiplexing gain tuples $(r_1, r_2, \ldots r_K)$.

• Theorem: If the block length $l \geq Km + n - 1$,

$$\mathcal{R}(d) = \left\{(r_1, r_2, \ldots r_K) : \sum_{s \in S} r_s \leq r^*_{|S|_{m,n}}(d), \forall S \subseteq 1, \ldots, K\right\}$$

where $r^*_{|S|_{m,n}}(\cdot)$ is the multiplexing-diversity tradeoff curve for a point-to-point channel with $|S|m$ transmit and $n$ receive antennas.
Conclusion

- We introduce the basic idea of multiple-antenna channels.

- Derive the capacity of the channels with different setup. The difficulty in evaluating channel capacity. Look for more practical measures to compare coding scheme.

- Diversity and multiplexing gain. The fundamental tradeoff.

- Extension to multiple-access channel.
References


