Homework 4

Exercise 5.13. 1. Let $\mathbf{1} = [1, 1, \cdots, 1]^T \in \mathcal{R}^L$. Then the channel equation is:

$$\mathbf{y} = \mathbf{1}x + \mathbf{z} \tag{5.7}$$

where $\mathbf{z} \sim CN(0, N_0\mathbf{I}_L)$ and x must satisfy the power constraint $E[x] \leq P$.

We note that we can project the received signal onto the direction of $\mathbf{1}$ obtaining the sufficient statistic:

$$r = \frac{\mathbf{1}^*}{\sqrt{L}}\mathbf{y} = \sqrt{L}x + \tilde{z} \tag{5.8}$$

where $\tilde{z} \sim CN(0, N_0)$. Defining $\tilde{x} = \sqrt{L}x$ we see that we have an AWGN channel with power constraint LP and noise variance N_0. Therefore $C = \log\left(1 + \frac{LP}{N_0}\right)$. We see that there is a power gain of L with respect to the single receive antenna system.

2. Let $\mathbf{h} = [h_1, h_2, \cdots, h_L]^T \in \mathcal{C}^L$. Then the channel equation is:

$$\mathbf{y} = \mathbf{h}x + \mathbf{z} \tag{5.9}$$

where $\mathbf{z} \sim CN(0, N_0\mathbf{I}_L)$, $\mathbf{h}$ is known at the receiver and x must satisfy the power constraint $E[x] \leq P$.

Since the receiver knows the channel, it can project the received signal onto the direction of $\mathbf{h}$ obtaining the sufficient statistic:

$$r = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{y} = \|\mathbf{h}\|x + \tilde{z} \tag{5.10}$$

where $\tilde{z} \sim CN(0, N_0)$. Then the problem reduces to computing the capacity of a scalar fading channel, with fading coefficient given by $\|\mathbf{h}\|$. It follows that:

$$C = E\left[\log\left(1 + \frac{\|\mathbf{h}\|^2 P}{N_0}\right)\right] = E\left[\log\left(1 + \frac{LP}{N_0}\frac{\|\mathbf{h}\|^2}{L}\right)\right] \tag{5.11}$$

In contrast, the single receive antenna system has a capacity $C = E\left[\log\left(1 + \frac{|h|^2 P}{N_0}\right)\right]$. The capacity is increased by having multiple receive antennas for two reasons:

first there is a power gain L, and second $\frac{\|\mathbf{h}\|^2}{L}$ has the same mean but less variance than $|h|^2$, and we get a diversity gain. Note that $Var\left[\frac{\|\mathbf{h}\|^2}{L}\right] = 1/L$ whereas $Var\left[|h|^2\right] = 1$.

As $L \to \infty$, $\frac{\|\mathbf{h}\|^2}{L} \to_{a.s.} 1$, so it follows that $C \approx \log\left(1 + \frac{LP}{N_0}\right)$ for large L.

3. With full CSI, the transmitter knows the channel, and for a given realization of the fading process $\{\mathbf{h}[n]\}_{n=1}^{N}$ the channel supports a rate:

$$R = \frac{1}{N}\sum_{n=1}^{N} \log\left(1 + \frac{\|\mathbf{h}[n]\|^2 P[n]}{N_0}\right) \tag{5.12}$$

and the problem becomes that of finding the optimal power allocation strategy. We note that the problem is the same as the one corresponding to the case of a single receive antenna, replacing $|h[n]|^2$ by $\|\mathbf{h}[n]\|^2$. It follows that the optimal solution is also obtained by waterfilling:

$$P^*(\|\mathbf{h}\|^2) = \left(\frac{1}{\lambda} - \frac{N_0}{\|\mathbf{h}\|^2}\right)^+ \tag{5.13}$$

where λ is chosen so that the power constraint is satisfied, i.e. $E[P^*(\|\mathbf{h}\|^2)] = P$. The resulting capacity is:

$$C = E\left[\log\left(1 + \frac{\|\mathbf{h}\|^2 P^*}{N_0}\right)\right] \tag{5.14}$$

At low SNR, when the system is power limited, the benefit of having CSI at the transmitter comes from the fact that we can transmit only when the channel is good, saving power (which is the limiting resource) when the channel is bad. The larger the fluctuation in the channel gain, the larger the benefit. If the channel gain is constant, then the waterfilling strategy reduces to transmitting with constant power, and there is no benefit in having CSI at the transmitter. When there are multiple receive antennas, there is diversity and $\|\mathbf{h}\|^2/L$ does not fluctuate much. In the limit as $L \to \infty$ we have seen that this random variable converges to a constant with probability one. Then, as L increases, the benefit of having CSI at the transmitter is reduced.

4.

$$P_{out} = Pr\left[\log\left(1 + \frac{\|\mathbf{h}\|^2 P}{N_0}\right) < R\right] = Pr\left[\|\mathbf{h}\|^2 < (2^R - 1)\frac{N_0}{P}\right] \tag{5.15}$$

We know that we can approximate the pdf of $\|\mathbf{h}\|^2$ around 0 by:

$$f(x) \approx \frac{1}{(L-1)!} x^{L-1} \tag{5.16}$$

where Rayleigh fading was assumed, and hence the distribution function of $\|\mathbf{h}\|^2$ evaluated at x is approximately given by:

$$F(x) \approx \frac{1}{L!} x^{L} \tag{5.17}$$

for x small. Thus, for large SNR we get the following approximation for the outage probability:

$$P_{out} \approx \frac{1}{L!} \left[(2^R - 1) \frac{N_0}{P} \right]^L \tag{5.18}$$

We see that having multiple antennas reduces the outage probability by a factor of $(2^R - 1)^L / L!$ and also increases the exponent of SNR^{-1} by a factor of L.

EXERCISE 8.23. We have the following sequence of steps:

$$
\begin{aligned}
p_{\text{out}}(R) &\overset{(a)}{\geq} \mathbb{P}\left\{\log\det\left(\mathbf{I}_{n_r} + \mathsf{SNR}\mathbf{H}\mathbf{H}^*\right) < R\right\}, \\
&\overset{(b)}{\geq} \mathbb{P}\left\{\mathsf{SNR}\mathrm{Tr}[\mathbf{H}\mathbf{H}^*] < R\right\}, \\
&\overset{(c)}{\geq} \mathbb{P}\left\{\mathsf{SNR}|h_{11}|^2 < \frac{R}{n_r n_t}\right\}^{n_r n_t}, \\
&\overset{(d)}{=} \left(1 - e^{-\frac{R}{n_r n_t \mathsf{SNR}}}\right)^{n_r n_t}, \\
&\overset{(e)}{\approx} \frac{R^{n_t n_R}}{(n_r n_t \mathsf{SNR})^{n_t n_r}}.
\end{aligned}
$$

Each of these steps can be justified as follows:

- (a): follows from letting each antenna power be SNR rather than SNR/n_t.
- (b): follows from the equation: $\mathsf{SNR}\mathrm{Tr}[\mathbf{H}\mathbf{H}^*] < \det\left(\mathbf{I}_{n_r} + \mathsf{SNR}\mathbf{H}\mathbf{H}^*\right)$ and hence a simple set theoretic containment relationship.
- (c): again follows from a simple set theoretic containment relationship:

  $$\left\{\mathsf{SNR}|h_{ij}|^2 < \frac{R}{n_r n_t} \;\forall\, i,j\right\} \subset \left\{\mathsf{SNR}\mathrm{Tr}[\mathbf{H}\mathbf{H}^*] < R \;\forall\, i,j\right\},$$

  and the facts that h_{ij}s are i.i.d.

- (d): follows from the fact $|h_{11}|^2$ is exponential.
- (e): follows from a simple Taylor series expansion.

Problem 2

Capacity with CSI only at Receiver:

$$C = \log_2\left(\det\left(\mathbf{I} + \frac{E_s}{N_0 M_T}\mathbf{H}\mathbf{H}^H\right)\right) \qquad (\because \mathbf{R}_{SS} = \mathbf{I})$$

SVD of the Channel Matrix:

$$\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^H = [\mathbf{U}_1 \mid \mathbf{U}_2]\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}[\mathbf{V}_1 \mid \mathbf{V}_2]^H$$

Capacity with CSI at both transmitter and Receiver:

$$C = \sum_{i=1}^{r} \log\left(1 + \sigma_i^2 \frac{E_s \gamma_i^{opt}}{N_0 M_T}\right)$$

Waterfilling:

$$\begin{aligned} \gamma_i^{opt} &= \left(\mu - \frac{M_T N_0}{E_S \sigma_i^2}\right)_+ \\ (X)_+ &= \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases} \\ \sum_{i=1}^{r} \gamma_i^{opt} &= M_T \end{aligned}$$

- Start with iteration counter p=1
- Calculate the constant μ

$$\mu = \frac{M_T}{r - p + 1}\left[1 + \frac{N_0}{E_S}\sum_{i=1}^{r-p+1}\frac{1}{\sigma_i^2}\right]$$

- Using μ calculate the power in the i^{th} sub-channel from,

$$\gamma_i = \left(\mu - \frac{M_T N_0}{E_S \sigma_i^2}\right) \qquad (i = 1, 2 \cdots r - p + 1)$$

- If the energy in the channel with lowest gain is negative, discard it by setting $\gamma_{r-p+1}^{opt} = 0$
- Repeat the procedure by setting, $p = p + 1$
- The optimal power allocation strategy is found when all the allocated power is non-negative

1. $\rho = 10 \text{ dB} \Longrightarrow \frac{E_s}{N_0} = 10$

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^H$$

$\sigma_1^2 = 2, \quad \sigma_2^2 = 2$

Since both the singular values are equal, we can allocate equal power to the two modes

We can see this even from the waterfilling procedure

- Start with iteration counter p=1
- Calculate the constant μ

$$\mu = \frac{M_T}{r-p+1}\left[1 + \frac{N_0}{E_S}\sum_{i=1}^{r-p+1}\frac{1}{\sigma_i^2}\right] = \frac{2}{2}\left[1 + \frac{1}{10}\sum_{i=1}^{2}\frac{1}{\sigma_i^2}\right] = \frac{11}{10}$$

- Using μ calculate the power in the i^th sub-channel from,

$$\begin{aligned}\gamma_i &= \left(\mu - \frac{M_T N_0}{E_S \sigma_i^2}\right) \quad (i = 1, 2 \cdots r-p+1)\\ \gamma_1 &= \left(\frac{11}{10} - \frac{2}{20}\right) = 1 = \gamma_2\end{aligned}$$

- Since both $\gamma_1,\ \gamma_1$ are not negative,$\gamma_1^{opt} = 1,\ \gamma_2^{opt} = 1$

(a) Capacity with CSI at both transmitter and Receiver:

$$C = \sum_{i=1}^{r}\log\left(1 + \sigma_i^2\frac{E_s\gamma_i^{opt}}{N_0 M_T}\right) = \sum_{i=1}^{2}\log\left(1 + \sigma_i^2\frac{10\gamma_i^{opt}}{2}\right) = 2\log_2(11) = 6.9189$$

(b) Capacity with CSI only at Receiver:

$$C = \log_2\left(\det\left(\mathbf{I} + \frac{E_s}{N_0 M_T}\mathbf{H}\mathbf{H}^H\right)\right) = 6.9189$$

Comment:
- Since our optimum power allocation strategy with CSI at transmitter gave $\mathbf{R}_{ss} = \mathbf{I}$(which is what we use for CSI unknown case), channel knowledge at the transmitter doesn't help in this case.

2. $\mathbf{H}$= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^H$

$\sigma_1^2 = 4, \quad \sigma_2^2 = 0$

$\mathbf{H}$ is rank deficient and so it has only one singular value. We should use only one mode and put all power into it. Waterfilling procedure is not required in this case.

(a) Capacity with CSI at both transmitter and Receiver:

$$C = \sum_{i=1}^{r}\log\left(1 + \sigma_i^2\frac{E_s\gamma_i^{opt}}{N_0 M_T}\right) = \log\left(1 + \sigma_i^2\frac{10\gamma_i^{opt}}{2}\right) = \log_2(41) = 5.3576$$

(b) Capacity with CSI only at Receiver:

$$C = \log_2 \left(\det \left(\mathbf{I} + \frac{E_s}{N_0 M_T} \mathbf{H}\mathbf{H}^H \right) \right) = 4.3923$$

Comments:
• Capacity values with or without CSI (at transmitter) are lower for this channel compared to the previous channel, this can be attributed to the rank deficient nature of this channel.

• The capacity values with and without CSI are different in this case, (For the previous channel they are same). This is due to the fact that we avoided putting energy into the mode that is in the nullspace, which we can't do if there is no CSI. As a consequence, the capacity without CSI is less than with CSI.

Problem 3

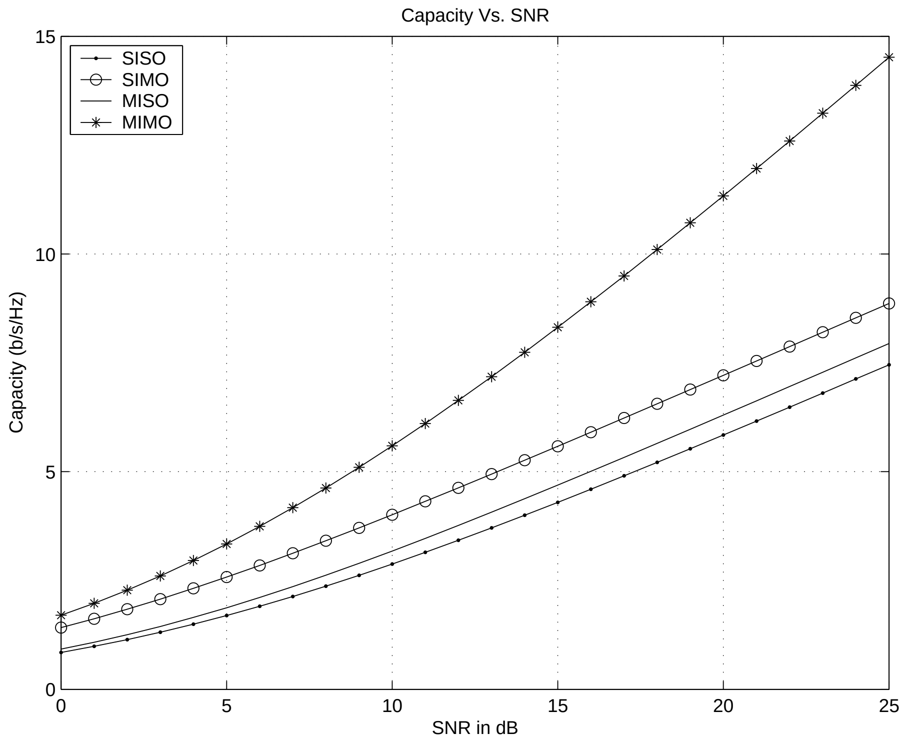

Figure 1: Capacity with different antenna configurations

At higher SNR's, Capacity increases linearly with the rank of channel matrix. The performance with SIMO is better than MISO. This is due to the array gain seen in SIMO case. Better performance of MIMO can be explained by the additional degrees of freedom available. All these plots assume that transmitter doesn't have CSI.

Capacity with CSI only at Receiver:

$$\begin{aligned} C_{MIMO} &= \log_2\left(\det\left(\mathbf{I} + \frac{\rho}{M_T}\mathbf{H}\mathbf{H}^H\right)\right) \\ \text{let, } r &= \min\{M_T, M_R\} \end{aligned}$$

Using a low SNR approximation, outage formulation is given by,

$$P\left\{\frac{\rho}{M_T} \parallel \mathbf{H} \parallel_F^2 \leq C_{MIMO}\right\} = x$$

$$P\left\{\frac{\parallel \mathbf{H} \parallel_F^2}{M_T M_R} \leq \frac{C_{SISO}}{\rho M_R}\right\} = x$$

Correspondingly for SISO,

$$P\left\{\parallel \mathbf{h} \parallel^2 \leq \frac{C_{MIMO}}{\rho}\right\} = x$$

Based on the tails of the distributions of '$\frac{\|\mathbf{H}\|_F^2}{M_T M_R}$' and '$\parallel \mathbf{h} \parallel^2$' we can see that to have the same outage (x), we need to integrate across more region in MIMO case compared to SISO. This leads to the following inequality,

$$C_{SISO} \leq \frac{C_{MIMO}}{M_R}$$

The above inequality is noticed to be valid from simulations also. A similar analysis can be carried out for high SNR regime, using the following approximation.

$$\begin{aligned} C_{MIMO} &= \log_2\left(\det\left(\mathbf{I} + \frac{\rho}{M_T}\mathbf{H}\mathbf{H}^H\right)\right) \\ &= r \log \rho + r \log \lambda \end{aligned}$$

Some observations from plots:

• For small ϵ's, and at high SNR's, the outage capacity ratio between MIMO and SISO is more than the ratio at higher outage values. Which means for smaller outage values having more antennas (at the transmitter or receiver) is more meaningful than at higher ϵ's.
• At moderate to high ϵ's, outage capacity follows the pattern of ergodic capacity, which is a linear growth in $\min\{M_T, M_R\}$
• At very high ϵ's (99%), outage capacity has interesting behavior. At moderate to high SNR's linear growth in $\min\{M_T, M_R\}$ is possible. But at very low SNR's, SIMO performs better than all other schemes. MISO has the worst performance of all, MIMO performs slightly better than SISO. This can be explained in terms of power distribution at the transmitter. At low SNR's it is better to use only one channel, since we don't have channel knowledge if we put energy in both modes the chance of it being in outage is pretty high. This could be one reason for better performance of SIMO compared to MIMO. MIMO performs slightly better than SISO because of the 2 receiving antennas.
P. 187-189 (Tse and Vishwanath) have an analytical treatment of outage capacity for SISO, SIMO and MISO cases.

Ergodic Capacity: MATLAB code - *Slight modification for outage code*

```
% mT, Number of transmitting antennas
% mR, Number of transmitting antennas
% ITER, number of trials
% SNRdB, Range of SNR in dB
% C_SISO, variable for capacity of a SISO system
% C_SIMO, variable for capacity of a SIMO system
% C_MISO, variable for capacity of a MISO system
% C_MIMO, variable for capacity of a MIMO system
% h_SISO, random channel for SISO (with zero mean unit variance)
% h_SIMO, random channel for SIMO
% h_MISO, random channel for MISO
% h_MIMO, random channel for MIMO
clc;
close all;
clear all;
mT = 2;
mR = 2;
ITER = 1000;
SNRdB = [0:25];
SNR = 10.^(SNRdB/10);
C_SISO = zeros(1,length(SNR));
C_SIMO = zeros(1,length(SNR));
C_MISO = zeros(1,length(SNR));
C_MIMO = zeros(1,length(SNR));
for ite = 1:ITER
h_SISO = (randn +j*randn)/sqrt(2);
h_SIMO = (randn(mR,1)+j*randn(mR,1))/sqrt(2);
h_MISO = (randn(1,mT)+j*randn(1,mT))/sqrt(2);
h_MIMO = (randn(mR,mT)+j*randn(mR,mT))/sqrt(2);
for K = 1:length(SNR)
C_SISO(K) = C_SISO(K) + log2(1+ SNR(K)*norm(h_SISO)^2);
C_SIMO(K) = C_SIMO(K) + log2(1+ SNR(K)*norm(h_SIMO)^2);
C_MISO(K) = C_MISO(K) + log2(1+ SNR(K)*norm(h_MISO)^2/mT);
C_MIMO(K) = C_MIMO(K) + log2(abs(det(eye(mR)+SNR(K)*h_MIMO*h_MIMO'/mT)));
end
end
C_SISO = C_SISO/ITER;
C_SIMO = C_SIMO/ITER;
C_MISO = C_MISO/ITER;
C_MIMO = C_MIMO/ITER;
plot(SNRdB,C_SISO,'r - .',SNRdB,C_SIMO,'b - o',SNRdB,C_MISO,'m',SNRdB,C_MIMO,'k - *')
legend('SISO','SIMO','MISO','MIMO',2)
xlabel('SNR in dB')
ylabel('Capacity (b/s/Hz)')
title('Capacity Vs. SNR')
grid;
```

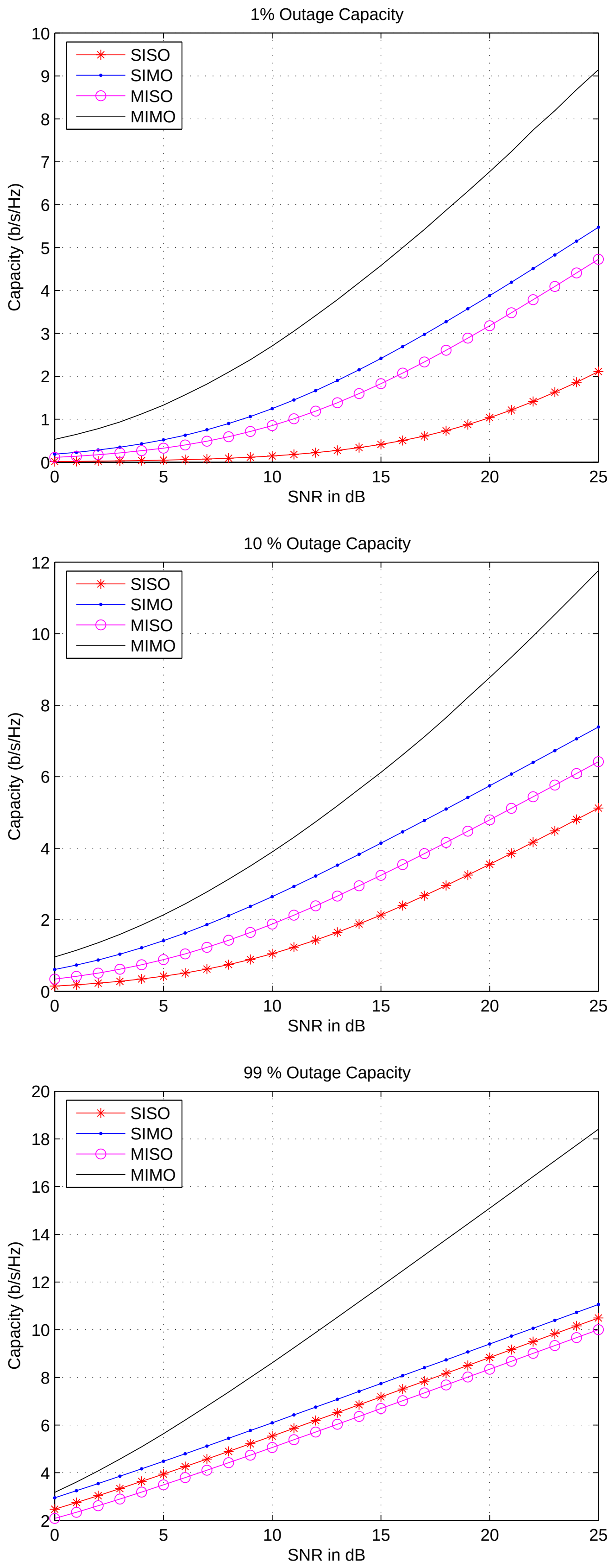

Figure 2: Outage Capacity: Increases linearly with the rank of channel matrix at high SNR's

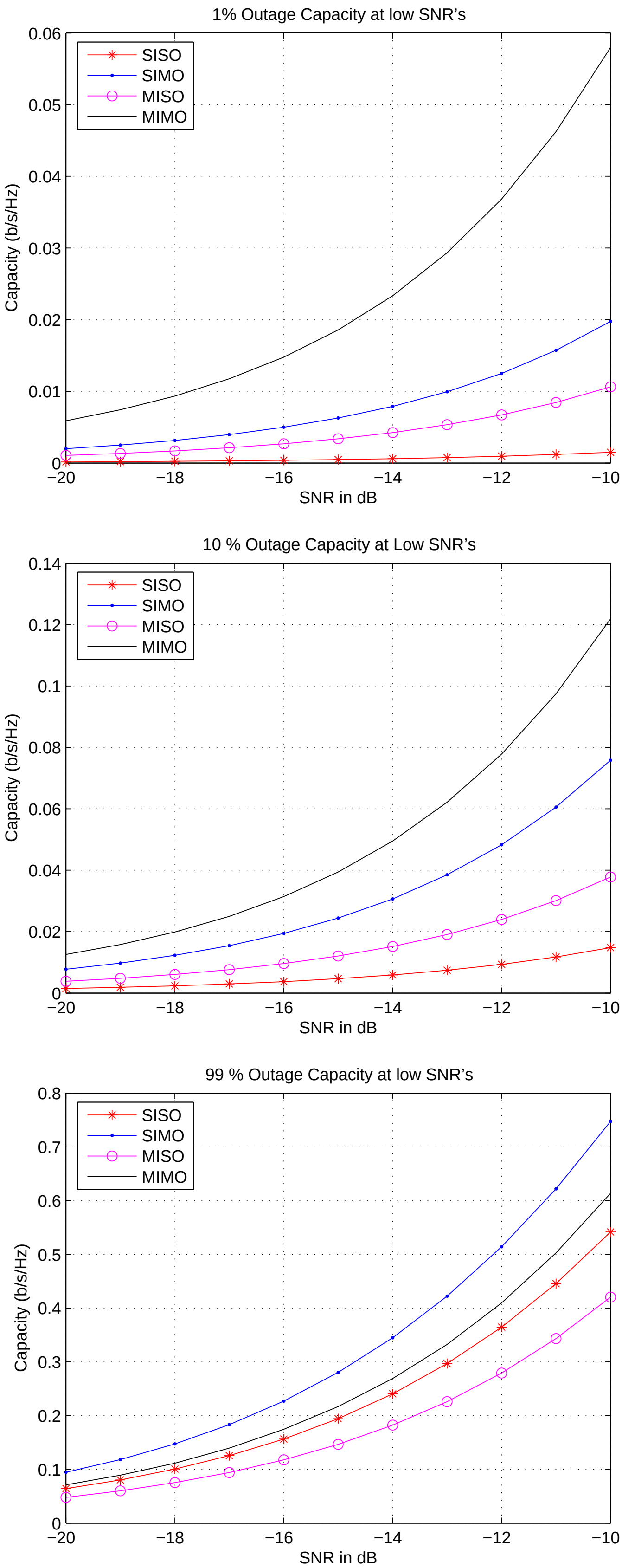

Figure 3: Outage Capacity at low SNR's