Midterm — ECE 275A — Fall 2009 — Solutions

[20 pts] **1.** As a consequence of the triangle inequality,[1] derive the "reverse triangle inequality"

$$\|x - z\| \geq |\, \|x\| - \|z\| \,| \, .$$

You *cannot* assume you are in an inner product space.

SOLUTION.

FIRST PROOF:

The triangle inequality gives $\|x + y\| \leq \|x\| + \|y\|$ for all x and y. Setting $z = x + y$, this is equivalent to

$$\|z\| - \|x\| \leq \|z - x\| = \|x - z\|$$

for all x and z. Because this is is true for all x and z, we can swap the roles of x and z, $x \rightleftarrows z$, to obtain

$$\|x\| - \|z\| \leq \|x - z\|$$

for all x and z. The two inequalities yield the desired result.

ALTERNATIVE PROOF:

For each x and z, $\|x\| = \|x - z + z\| \leq \|x - z\| + \|z\|$, or

$$\|x - z\| \geq \|x\| - \|z\| \, .$$

On the other hand, $\|z\| = \|z - x + x\| \leq \|z - x\| + \|x\| = \|x - z\| + \|x\|$, so that

$$\|x - z\| \geq \|z\| - \|x\| \, .$$

Again, the two inequalities yield the desired result.

[1] Indeed, the Wikipedia website on the triangle inequality calls it "an elementary consequence" of the triangle inequality. The derivation is not hard or lengthy.

[80 pts] **2. BACKGROUND.**

Let $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ for $t \in \mathbf{T} = [0, 1]$. The state $x(t)$ evolves on $\mathbf{T}$ as

$$\boxed{\Sigma: \quad \dot{x} = Ax + Bu \quad \text{with} \quad x(0) = 0\,.} \tag{1}$$

With zero initial condition, $x(0) = 0$, the solution to the above dynamical system Σ is

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \quad \text{for} \quad t \in \mathbf{T}\,,$$

where the *state transition matrix* e^{At} is defined by the matrix Taylor series expansion

$$e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots$$

Term-by-term manipulation in the expansion shows that $\left(e^{At}\right)^T = e^{A^T t}$.

In particular, we have a linear mapping of $u(t)$, $t \in \mathbf{T}$, to $x(1) \in \mathbb{R}^n$,

$$\boxed{x(1) = \mathcal{A}(u) \triangleq \int_0^1 e^{A(1-\tau)} B u(\tau) d\tau\,,}$$

$\mathbb{R}^n$ is taken to be the standard n-dimensional inner product space, $\langle x(1), z(1) \rangle = x(1)^T z(1)$. The function u is assumed to belong to the (infinite dimensional) Hilbert space of real-valued square-integrable (i.e., *finite energy*) functions, $u \in L_2 = L_2[0, 1]$,

$$\boxed{\mathsf{Energy}(u) \triangleq \int_0^1 u^2(\tau) d\tau < \infty \quad \text{and} \quad \langle u_1, u_2 \rangle = \int_0^1 u_1(\tau) u_2(\tau) d\tau\,.}$$

We have, then, a linear mapping between two Hilbert spaces,

$$\boxed{\mathcal{A} : L_2 \to R^n \quad \text{given by} \quad u \mapsto x(1) = \mathcal{A}(u)\,.}$$

The system (1) is <u>*completely controllable*</u> (CC) iff the *controllability matrix*

$$\boxed{P_c \triangleq [B, AB, A^2 B, \cdots, A^{n-1} B]}$$

has full rank which, in turn, is true iff the *controllability grammian*, $G(1)$, is full rank,[2]

$$\boxed{G(1) \triangleq \int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)} d\tau\,.}$$

[2] Thus Σ being CC, P_c having full rank, and G having full rank are all *equivalent* statements.

(a) Derive an algebraic condition for $\mathcal{A}$ to be onto. Given your derived condition, what can we say about the system Σ if $\mathcal{A}$ is onto?

SOLUTION.

$\mathcal{A}$ is onto iff $\mathcal{A}^*$ is one-to-one, which is true iff $\mathcal{A}\mathcal{A}^*$ is one-to-one. We have[3]

$$\begin{aligned}
\langle x(1), \mathcal{A}(u)\rangle &= x^T \int_0^1 e^{A(1-\tau)} B u(\tau) d\tau \\
&= \int_0^1 x^T e^{A(1-\tau)} B u(\tau) d\tau \\
&= \int_0^1 \left[e^{A(1-\tau)} B\right]^T x(1) u(\tau) d\tau \\
&= \int_0^1 \left[B^T e^{A^T(1-\tau)} x(1)\right] u(\tau) d\tau \\
&= \int_0^1 \left[\, \mathcal{A}^*(x(1))(\tau)\, \right] u(\tau) d\tau \\
&= \langle \mathcal{A}^*(x(1)), u\rangle \, ,
\end{aligned}$$

showing that $\mathcal{A}^*$ acts on any vector $x(1) \in \mathbb{R}^n$ to produce a vector $v = \mathcal{A}^*(x(1)) \in L_2[0,1]$ via

$$v(t) = \mathcal{A}^*(x(1))(t) = B^T e^{A^T(1-t)} x(1) \, , \quad t \in \mathbf{T} = [0,1] \, , \tag{2}$$

which we can write informally as

$$\mathcal{A}^* = B^T e^{A^T(1-t)} \, , \quad t \in \mathbf{T} \, .$$

This yields

$$\mathcal{A}\mathcal{A}^*(x(1)) = \underbrace{\int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)} d\tau}_{G(1) \;=\; \mathcal{A}\mathcal{A}^*} \; x(1) \, . \tag{3}$$

Thus, the system Σ is onto iff the controllability grammian $G(1) = \mathcal{A}\mathcal{A}^*$ is full rank (is nonsingular), which is true iff the controllability matrix P_c is full rank, which is true iff the system is CC.

Practically, this means that if the system Σ is CC, a control can be found that will drive the system to any state $x(1)$ at time $t = 1$ from the origin at time $t = 0$. *A simple algebraic test for the ability to do this is to check that $P_c \in R^{n \times n}$ is nonsingular.*

[3] Using the facts that $x^T y = y^T x$ and $\left(e^{At} B\right)^T = B^T e^{A^T t}$.

(b) For an *arbitrary* target state $x(1)$, derive a *minimum energy control* $u \in L_2$ that will drive the system to the arbitrary target state. You are allowed to make all necessary assumptions needed to do this problem, but you must *clearly state* just what your assumptions are.

SOLUTION.

To drive the system to an *arbitrary state* $x(1)$ it is necessary and sufficient that $\mathcal{A}$ be onto, which we have shown in the previous problem is equivalent to the system being CC. Thus we make the necessary and sufficient assumption that the system is CC, which ensures that the controllability grammian $G(1) = \mathcal{A}\mathcal{A}^*$ is full rank (and hence invertible). Noting that the minimum energy solution is equal to the minimum norm solution on $L_2[0,1]$, we can immediately use the results of the previous problem to obtain

$$u = \mathcal{A}^+ x(1) = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1} x(1)\,, \tag{4}$$

as

$$u(t) = B^T e^{A^T(1-t)}\, G^{-1}(1)\, x(1)\,, \tag{5}$$

with

$$G(1) \triangleq \int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)} d\tau\,. \tag{6}$$

(c) Consider an axially rotational system that obeys the rotational dynamics

$$J\ddot{\theta} + D\dot{\theta} = T\,. \tag{7}$$

(i) Taking $u = T/J \in L_2[0,1]$ and $D/J = 1$ and defining a state vector x whose components are the "phase variables" $x_1 = \theta$, $x_2 = \dot{\theta}$, place this system into the state-space form of Eq. (1). (ii) Determine if the resulting system is completely controllable (CC). (iii) What is the practical consequence of your conclusion?

SOLUTION.

Our dynamical equation of interest is $\ddot{\theta} + \dot{\theta} + u$. Note that it is immediately apparent that

$$\dot{x_1} = x_2$$

while

$$\dot{x_2} = \ddot{\theta} = -\dot{\theta} + u = -x_2 + u\,.$$

These two equations yield

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u\,.$$

The controllability matrix,

$$P_c = (B, AB) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} ,$$

is nonsingular so the rotational system is CC. Practically, this means that a control exists that can drive the system to any desired state at time $t = 1$ from the origin at time $t = 0$ and in particular, as was shown above, a minimum energy control can be found to do the job.

(d) If A can be diagonalized as $A = Q\Lambda Q^{-1}$, then it is easily shown from the Taylor series expansion for e^{At} that

$$e^{At} = e^{Q\Lambda t Q^{-1}} = Q\, e^{\Lambda t}\, Q^{-1} = Q\,\mathrm{Diag}\left(e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\right) Q^{-1} .$$

It is a fact that if A has n *distinct* eigenvalues $\lambda_i \neq \lambda_j$, $i \neq j$, then A is *guaranteed* to have n *linearly independent eigenvectors* v_i, $Av_i = \lambda v_i$, and therefore $Q = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ is guaranteed to be nonsingular (and hence invertible). Thus a matrix A is diagonalizable when its eigenvalues are distinct.[4]

With this background, do the following:

i. For the system (7), find an explicit form of the adjoint of the linear operator that maps u to $x(1)$.

SOLUTION.

To compute the adjoint operator using the formula shown in Eq. (2) we must first determine e^{At}. This requires that we find the eigenvalues and eigenvectors of A. The eigenvalues of A are the solutions to

$$\det(sI - A) = s(s+1) = 0 ,$$

which are $\lambda_1 = -1, \lambda_2 = 0$.[5] The eigenvectors v_1 and v_2 of A are determined by[6]

$$(\lambda_1 I - A)\, v_1 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} v_1 = 0 \implies v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$(\lambda_2 I - A)\, v_2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} v_2 = 0 \implies v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

This yields

$$Q = (v_1\ v_2) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad Q^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} .$$

[4] A general matrix is *not* quaranteed to be diagonalizable as it may have a shortage of linearly independent eigenvectors.

[5] Or *vice versa*—the naming order, of course, is arbitrary.

[6] Because A is in so-called "lower companion form" its eigenvectors are actually trivial to compute directly from the eigenvalues. However, I don't expect students to know this fact, so I gave you a problem with a system matrix that has eigenvectors which can be determined easily by inspection using the condition $(\lambda_i I - A)v_i = 0$. (Which, of course, is equivalent to $Av_i = \lambda_i v_i$.)

Thus

$$e^{At} = Q\,\mathsf{Diag}\left(e^{\lambda_1 t},\, e^{\lambda_2 t}\right) Q^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

yielding

$$e^{At}B = \begin{pmatrix} 1-e^{-t} \\ e^{-t} \end{pmatrix} \quad \text{and} \quad B^T e^{A^T t} = \left(e^{At}B\right)^T = \left(1-e^{-t}\,,\;\; e^{-t}\right) \tag{8}$$

From Eq. (2) we obtain the adjoint operator $\mathcal{A}^*$, which acts on any 2–vector $\xi \in \mathbb{R}^2$ according to

$$\mathcal{A}^*(\xi)(t) = B^T e^{A^T(1-t)}\xi = \left(1-e^{t-1}\,,\;\; e^{t-1}\right)\xi = \left(1-e^{t-1}\right)\xi_1 + e^{t-1}\,\xi_2\,. \tag{9}$$

We can informally write the adjoint operator as

$$\mathcal{A}^* = \left(1-e^{t-1}\,,\;\; e^{t-1}\right) = \frac{1}{e}\left(e-e^t\,,\;\; e^t\right)\,, \quad t \in \mathbf{T} = [0,1]\,. \tag{10}$$

ii. Derive the minimum energy control law that takes the system from position $\theta(0) = 0$ and $\dot\theta = 0$ to position $\theta(1) = 1$ and $\dot\theta(1) = 0$.[7]

<u>SOLUTION</u>.

The solution is given by Eq.'s (4)–(6) once we have found the adjoint operator $\mathcal{A}^*$ and the controllability grammian $G(1) = \mathcal{A}\mathcal{A}^*$. We have already found the adjoint operator in the previous problem. It remains to determine $G(1)$ and $G^{-1}(1)$. From Eq.'s (3) and (8) we have

$$\begin{aligned} G(1) &= \int_0^1 e^{A(1-t)}BB^T e^{A^T(1-t)}dt \\ &= \frac{1}{e^2}\int_0^1 \begin{pmatrix} e-e^t \\ e^t \end{pmatrix}\left(e-e^t\,,\;\; e^t\right)dt \\ &= \frac{1}{e^2}\int_0^1 \begin{pmatrix} \left(e-e^t\right)^2 & e^t\left(e-e^t\right) \\ e^t\left(e-e^t\right) & e^{2t} \end{pmatrix} dt \\ &= \frac{1}{e^2}\int_0^1 \begin{pmatrix} e^2-2ee^t+e^{2t} & ee^t-e^{2t} \\ ee^t-e^{2t} & e^{2t} \end{pmatrix} dt\,. \end{aligned}$$

To integrate each component of the matrix, recall that

$$\int_0^1 e^{\alpha t}dt = \frac{1}{\alpha}\,e^{\alpha t}\Big|_0^1 = \frac{1}{\alpha}\left(e^\alpha - 1\right)$$

[7] The answer is $u^{\rm opt} = \frac{1+e-2e^t}{3-e}$. I'm mainly interested in whether or not you know what you are doing so don't worry too much about getting the exact answer. The derivation *is* tedious.

showing that under the action of the integration we have

$$e^t \to e - 1 \quad \text{and} \quad e^{2t} \to \frac{1}{2}\left(e^2 - 1\right) \; .$$

Continuing,

$$\begin{aligned}
G(1) &= \frac{1}{e^2}\begin{pmatrix} e^2 - 2e(e-1) + \frac{1}{2}(e^2-1) & e(e-1) - \frac{1}{2}(e^2-1) \\ e(e-1) - \frac{1}{2}(e^2-1) & \frac{1}{2}(e^2-1) \end{pmatrix} \\
&= \frac{1}{2e^2}\begin{pmatrix} 2e^2 - 4e(e-1) + (e^2-1) & 2e(e-1) - (e^2-1) \\ 2e(e-1) - (e^2-1) & (e^2-1) \end{pmatrix} \\
&= \frac{1}{2e^2}\begin{pmatrix} -e^2 + 4e - 1 & e^2 - 2e + 1 \\ e^2 - 2e + 1 & e^2 - 1 \end{pmatrix}
\end{aligned}$$

finally yielding

$$G(1) = \frac{1}{2e^2}\begin{pmatrix} -e^2 + 4e - 1 & (e-1)^2 \\ (e-1)^2 & (e^2-1) \end{pmatrix} \; . \tag{11}$$

The inverse of $G(1)$ is given by

$$G^{-1}(1) = \frac{\text{Adj}(G(1))}{\det G(1)}$$

where the *adjugate matrix* Adj$(G(1))$ is the transpose of the matrix of cofactors of $G(1)$. Adj$(G(1))$ is, of course, straightforward to compute for a 2×2 matrix. We have

$$\text{Adj}(G(1)) = \frac{1}{2e^2}\begin{pmatrix} (e^2-1) & -(e-1)^2 \\ -(e-1)^2 & -e^2 + 4e - 1 \end{pmatrix} \; .$$

We also have (recalling that $e^2 - 1 = (e+1)(e-1)$)

$$\begin{aligned}
\det G(1) &= \left(\frac{1}{2e^2}\right)^2 (e-1)\left[(e+1)(4e - e^2 - 1) - (e-1)^3\right] \\
&= \left(\frac{1}{2e^2}\right)^2 (e-1)\left[\,2\,e^2\,(3-e)\right] \\
&= \frac{1}{2e^2}(e-1)(3-e)\,.
\end{aligned}$$

Therefore

$$G^{-1}(1) = \frac{1}{(e-1)(3-e)}\begin{pmatrix} (e^2-1) & -(e-1)^2 \\ -(e-1)^2 & 4e - e^2 - 1 \end{pmatrix} \; . \tag{12}$$

With Eq. (10) we have

$$\mathcal{A}^+ = \mathcal{A}^* G^{-1}(1) = \frac{\left(e - e^t\,, \;\; e^t\right)}{e(e-1)(3-e)}\begin{pmatrix} (e^2-1) & -(e-1)^2 \\ -(e-1)^2 & 4e - e^2 - 1 \end{pmatrix} \; . \tag{13}$$

With $x(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the final optimal (minimum energy) control is given by

$$
\begin{aligned}
u^{\mathsf{opt}} &= \mathcal{A}^{+} x(1) = \mathcal{A}^{+} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\big(e - e^t\,, \quad e^t\big)}{e(e-1)(3-e)} \begin{pmatrix} (e^2 - 1) \\ -(e-1)^2 \end{pmatrix} = \frac{\big(e - e^t\,, \quad e^t\big)}{e(e-1)(3-e)} \begin{pmatrix} (e-1)(e+1) \\ -(e-1)(e-1) \end{pmatrix} \\
&= \frac{\big(e - e^t\,, \quad e^t\big)}{e(3-e)} \begin{pmatrix} 1+e \\ 1-e \end{pmatrix} = \frac{(e-e^t)(1+e) + e^t(1-e)}{e(3-e)}
\end{aligned}
$$

or

$$
u^{\mathsf{opt}}(t) = \frac{1 + e - 2e^t}{3 - e}\,, \quad t \in \mathbf{T} = [0, 1]\,. \tag{14}
$$