Midterm — ECE 275A — Fall 2009 — 100 Points Total

[20 pts] **1.** As a consequence of the triangle inequality,[1] derive the "reverse triangle inequality"

$$\|x - z\| \geq |\, \|x\| - \|z\| \,| \, .$$

You *cannot* assume you are in an inner product space.

[80 pts] **2.** Let $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ for $t \in \mathbf{T} = [0, 1]$. The state $x(t)$ evolves on $\mathbf{T}$ as

$$\boxed{\Sigma : \quad \dot{x} = Ax + Bu \quad \text{with} \quad x(0) = 0 \, .} \tag{1}$$

With zero initial condition, $x(0) = 0$, the solution to the above dynamical system Σ is

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \quad \text{for} \quad t \in \mathbf{T} \, ,$$

where the *state transition matrix* e^{At} is defined by the matrix Taylor series expansion

$$e^{At} = I + At + \frac{1}{2!}A^2 t^2 + \frac{1}{3!}A^3 t^3 + \cdots$$

Term-by-term manipulation in the expansion shows that $\left(e^{At}\right)^T = e^{A^T t}$.

In particular, we have a linear mapping of $u(t)$, $t \in \mathbf{T}$, to $x(1) \in \mathbb{R}^n$,

$$\boxed{x(1) = \mathcal{A}(u) \triangleq \int_0^1 e^{A(1-\tau)} Bu(\tau) d\tau \, ,}$$

$\mathbb{R}^n$ is taken to be the standard n-dimensional inner product space, $\langle x(1), z(1) \rangle = x(1)^T z(1)$. The function u is assumed to belong to the (infinite dimensional) Hilbert space of real-valued square-integrable (i.e., *finite energy*) functions, $u \in L_2 = L_2[0, 1]$,

$$\boxed{\mathsf{Energy}(u) \triangleq \int_0^1 u^2(\tau) d\tau < \infty \quad \text{and} \quad \langle u_1, u_2 \rangle = \int_0^1 u_1(\tau) u_2(\tau) d\tau \, .}$$

We have, then, a linear mapping between two Hilbert spaces,

$$\boxed{\mathcal{A} : L_2 \to R^n \quad \text{given by} \quad u \mapsto x(1) = \mathcal{A}(u) \, .}$$

The system (1) is <u>*completely controllable*</u> (CC) iff the *controllability matrix*

$$\boxed{P_c \triangleq [B, AB, A^2 B, \cdots, A^{n-1} B]}$$

has full rank which, in turn, is true iff the *controllability grammian*, $G(1)$, is full rank,[2]

$$\boxed{G(1) \triangleq \int_0^1 e^{A(1-\tau)} B B^T e^{A^T (1-\tau)} d\tau \, .}$$

[1] Indeed, the Wikipedia website on the triangle inequality calls it "an elementary consequence" of the triangle inequality. The derivation is *subtle,* not hard or lengthy.

[2] Thus Σ being CC, P_c having full rank, and G having full rank are all *equivalent* statements.

(a) Derive an algebraic condition for $\mathcal{A}$ to be onto. Given your derived condition, what can we say about the system Σ if $\mathcal{A}$ is onto?

(b) For an *arbitrary* target state $x(1)$, derive a *minimum energy control* $u \in L_2$ that will drive the system to the arbitrary target state. You are allowed to make all necessary assumptions needed to do this problem, but you must *clearly state* just what your assumptions are.

(c) Consider an axially rotational system that obeys the rotational dynamics

$$J\ddot{\theta} + D\dot{\theta} = T\,. \tag{2}$$

(i) Taking $u = T/J \in L_2[0,1]$ and $D/J = 1$ and defining a state vector x whose components are the "phase variables" $x_1 = \theta$, $x_2 = \dot{\theta}$, place this system into the state-space form of Eq. (1). (ii) Determine if the resulting system is completely controllable (CC). (iii) What is the practical consequence of your conclusion?

(d) If A can be diagonalized as $A = Q\Lambda Q^{-1}$, then it is easily shown from the Taylor series expansion for e^{At} that

$$e^{At} = e^{Q\Lambda t Q^{-1}} = Q\, e^{\Lambda t}\, Q^{-1} = Q\,\mathrm{Diag}\left(e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\right) Q^{-1}\,.$$

It is a fact that if A has n *distinct* eigenvalues $\lambda_i \neq \lambda_j$, $i \neq j$, then A is *guaranteed* to have n *linearly independent eigenvectors* v_i, $Av_i = \lambda v_i$, and therefore $Q = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ is guaranteed to be nonsingular (and hence invertible). Thus a matrix A is diagonalizable when its eigenvalues are distinct.[3]

With this background, do the following:

i. For the system (2), find an explicit form of the adjoint of the linear operator that maps u to $x(1)$.

ii. Derive the minimum energy control law that takes the system from position $\theta(0) = 0$ and $\dot{\theta} = 0$ to position $\theta(1) = 1$ and $\dot{\theta}(1) = 0$.[4]

Clearly state and show each step of your derivation.[5] I will be generous with partial credit iff I am convinced you are progressing in a principled, logical, step-by-step way towards obtaining the solution.

[3] A general matrix is *not* quaranteed to be diagonalizable as it may have a shortage of linearly independent eigenvectors.

[4] The answer is $u^{\mathrm{opt}} = \frac{1+e-2e^t}{3-e}$. I'm mainly interested in whether or not you know what you are doing so don't worry too much about getting the exact answer. The derivation *is* tedious.

[5] To manage the complexity of the problem, only compute quantities that *directly* relate to the answer.