ECE 174 Fall 2017
Supplemental Solutions to Homework 3

1. It is obvious that the rank of the matrix is 2 (as the two rows and the first two columns are linearly independent). The two linearly independent rows span the row space (i.e., they form a basis for the row space), which is obviously 2 dimensional. The first two, linearly independent columns, span the column space (i.e., they form a basis for the column space), which is obviously 2 dimensional. The nullspace is obviously 1 dimensional and spanned by the canonical coordinate vector $e_1^T (0\ 0\ 1)^T$ so that a basis for the nullspace is given by $\{e_1\}$.

Meyer 5.1.8 (a)   • To show $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$, we have,

$$\|\mathbf{x}\|_1^2 = \left(\sum_{i=1}^{n} |x_i| \right)^2 = \sum_{i=1}^{n} |x_i|^2 + \sum_{i \neq j} |x_i||x_j| = \|\mathbf{x}\|_2^2 + C$$

where $C \geq 0$. So $\|\mathbf{x}\|_1^2 \geq \|\mathbf{x}\|_2^2$, i.e. $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$.

• To show $\|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$, we have,

$$\|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} |x_i|^2 \geq \max_j |x_j|^2 = \|\mathbf{x}\|_\infty^2$$

so $\|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$.

(b)   • To show $\|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2$, we have,

$$\|\mathbf{x}\|_1 = \mathbf{e}^T |\mathbf{x}| \leq \|\mathbf{e}\|_2 \|\mathbf{x}\|_2 = \sqrt{n}\,\|\mathbf{x}\|_2$$

where $\mathbf{e}$ is the vector of all 1's, and $|\mathbf{x}|$ is the vector whose ith component is $|x_i|$.

• To show $\|\mathbf{x}\|_2 \leq \sqrt{n}\,\|\mathbf{x}\|_\infty$, we have,

$$\|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} |x_i|^2 \leq \sum_{i=1}^{n} \max_j |x_j|^2 = n \max_j |x_j|^2 = n \|\mathbf{x}\|_\infty^2$$

so $\|\mathbf{x}\|_2 \leq \sqrt{n}\,\|\mathbf{x}\|_\infty$.

- To show $\|\mathbf{x}\|_1 \leq n \|\mathbf{x}\|_\infty$, we have,

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} \max_j |x_j| = n \max_j |x_j| = n \|\mathbf{x}\|_\infty .$$

8. Recall that we (and Meyer) define the inner product to be linear in the second argument.[1] Also recall that $\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$.

(a) $\langle x_1, \alpha x_2 \rangle = \alpha \langle x_1, x_2 \rangle = \overline{\overline{\alpha} \overline{\langle x_1, x_2 \rangle}} = \overline{\overline{\alpha} \langle x_2, x_1 \rangle} = \overline{\langle x_2, \overline{\alpha} x_1 \rangle} = \langle \overline{\alpha} x_1, x_2 \rangle$.

(b) $\langle \alpha_1 x_1 + \alpha_2 x_2, x \rangle = \overline{\langle x, \alpha_1 x_1 + \alpha_2 x_2 \rangle} = \overline{\alpha_1 \langle x, x_1 \rangle + \alpha_2 \langle x, x_2 \rangle}$
$= \overline{\alpha}_1 \langle x_1, x \rangle + \overline{\alpha}_2 \langle x_2, x \rangle$

Note that in part (b) we obtain the property of *additivity in the first argument* as a special case (just take $\alpha_1 = \alpha_2 = 1$).

9. Recall that the operators $A : \mathcal{X} \to \mathcal{Y}$, $B : \mathcal{X} \to \mathcal{Y}$, and $C : \mathcal{Y} \to \mathcal{Z}$ are linear and that the adjoint operator, A^*, is defined by

$$\langle A^* x_1, x_2 \rangle = \langle x_1, A x_2 \rangle .$$

You also need to recall the properties of the inner product (such as linearity in the second argument) and the additional properties proved in Problem 1 above.

(a) $\langle y, \alpha A x \rangle = \langle y, A \alpha x \rangle = \langle A^* y, \alpha x \rangle = \langle \overline{\alpha} A^* y, x \rangle$.

(b) $\langle y, (A + B)x \rangle = \langle y, Ax + Bx \rangle = \langle y, Ax \rangle + \langle y, Bx \rangle = \langle A^* y, x \rangle + \langle B^* y, x \rangle$
$\langle A^* y + B^* y, x \rangle = \langle (A^* + B^*) y, x \rangle$.

(c) The fact that $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$ follows immediate from properties (b) and (a), in that order.

(d) $\langle x, A^* y \rangle = \overline{\langle A^* y, x \rangle} = \overline{\langle y, Ax \rangle} = \langle Ax, y \rangle$.

(e) $\langle z, CAx \rangle = \langle C^* z, Ax \rangle = \langle A^* C^* z, x \rangle$.

(f) $\langle A^* \alpha y, x \rangle = \langle \alpha y, Ax \rangle = \overline{\alpha} \langle y, Ax \rangle = \overline{\alpha} \langle A^* y, x \rangle = \langle \alpha A^* y, x \rangle$.

(g) $\langle A^*(y_1 + y_2), x \rangle = \langle y_1 + y_2, Ax \rangle = \langle y_1, Ax \rangle + \langle y_2, Ax \rangle = \langle A^* y_1, x \rangle + \langle A^* y_2, x \rangle$
$= \langle A^* y_1 + A^* y_2, x \rangle$.

(h) Linearity of A^* follows from properties (g) and (f), in that order.

10. Recall that the inner–products are defined as,

$$\langle x_1, x_2 \rangle = x_1^H \Omega x_2 \quad \text{and} \quad \langle y_1, y_2 \rangle = y_1^H W y_2 ,$$

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[1] Whereas many other authors define the inner product to be linear in the first argument. Of course, in the real vector space case the inner product is linear in both arguments so that the distinction disappears.

where Ω and W are hermitian (i.e., $\Omega^H = \Omega$ and $W^H = W$) and positive–definite (and hence both are invertible). Note that in this case their inverses are also hermitian, positive–definite.

(a) $\langle y, Ax \rangle = y^H W A x = y^H W A \Omega^{-1} \Omega x = \left((\Omega^{-1} A^H W) y \right)^H \Omega x = \langle (\Omega^{-1} A^H W) y, x \rangle$.

(b) For $r(A) = n$, A has *full column rank* and it must be the case that $m \geq n$. Since A is possibly over–determined, we solve the least squares problem by enforcing the geometric condition that $y - Ax \in \mathcal{R}(A)^\perp = \mathcal{N}(A^*)$. This yields the normal equations,

$$A^* Ax = A^* y.$$

Because $r(A) = n$, the $n \times n$ matrix $A^* A$ also has rank n and is therefore invertible. (This fact is consistent with the nullspace of A being trivial, so that the least–squares problem must have a unique solution.) Thus, we have that

$$r(A) = n \Rightarrow \widehat{x} = (A^* A)^{-1} A^* y,$$

for any value of y. It must therefore be the case that

$$r(A) = n \Rightarrow A^+ = (A^* A)^{-1} A^*,$$

where $A^* = \Omega^{-1} A^H W$ as determined in Part (a). With W a full rank hermitian matrix and $r(A) = n$, it is the case that $A^H W A$ is invertible and as a consequence the pseudoinverse, A^+, is *independent* of the weighting matrix Ω,

$$r(A) = n \Rightarrow A^+ = (A^H W A)^{-1} A^H W.$$

(c) For $r(A) = m$, A has full row–rank and therefore it must be the case that $n \geq m$. Note that A is *onto*, and therefore $y = Ax$ is solvable for all y. However, the system is possibly underdetermined, so we want to look for a minimum norm solution. This requires that we enforce the constraint that any solution to $y = Ax$ must also satisfy the geometric condition that $x \in \mathcal{N}(A)^\perp = \mathcal{R}(A^*)$. This condition is equivalent to,
$$x = A^* \lambda,$$
for some vector λ.

This condition, together with the requirement that x be a solution to $y = Ax$, yields,
$$AA^* \lambda = y.$$

Because $r(A) = m$, the $m \times m$ matrix AA^* also has rank n and is invertible. Thus,
$$\lambda = (AA^*)^{-1} y$$

which yields the result that

$$r(A) = m \Rightarrow \widehat{x} = A^*(AA^*)^{-1}y\,,$$

for all y. Thus,

$$r(A) = m \Rightarrow A^+ = A^*(AA^*)^{-1}\,.$$

With $r(A) = m$ and Ω^{-1} a hermitian full–rank matrix, it is the case that the $m \times m$ matrix $A\Omega^{-1}A^H$ is invertible. With the fact that $A^* = \Omega^{-1}A^H W$, this yields the fact that for $r(A) = m$, the pseudoinverse is *independent* of the weighting matrix W,

$$r(A) = m \Rightarrow A^+ = \Omega^{-1}A^H(A\Omega^{-1}A^H)^{-1}\,.$$

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