1. Let the \(m \times n\) matrix \(A\) be a linear mapping between two complex Hilbert spaces \(\mathcal{X}\) and \(\mathcal{Y}\) with inner-product weighting matrices given by \(\Omega\) and \(W\) respectively.

(a) Derive the form of the adjoint of \(A\), from the fundamental definition of the adjoint.

(b) Consider the inverse problem \(y = Ax\). i) Derive the algebraic condition for a least-squares solution to exist. ii) Derive the algebraic condition for a least-squares solution to be the minimum norm solution.

(c) Derive the pseudoinverse of \(A\) for the following two conditions. i) \(A\) has full column rank. ii) \(A\) has full row rank.

(d) Construct the pseudoinverse of \(A\) for \(\Omega\), \(W\), and \(A\) given by

\[
\Omega = \begin{pmatrix}
4.80 & 1 + 2j & 0.01 + 0.02j \\
1 - 2j & 9.76 & 2 - j \\
0.01 - 0.02j & 2 + j & 129.21
\end{pmatrix}, \quad
W = \begin{pmatrix}
721.839 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \(j = \sqrt{-1}\). Give the pseudoinverse in its simplest possible form.

Solution:

The solutions to parts (a), (b), and (c) can be found in the lecture and in the solutions to homework 3. For part (d), because the matrix has full column rank the pseudoinverse is independent of \(\Omega\) and has the form (see homework assignment 3 and note that \(A\) is real)

\[
A^+ = (A^HWA)^{-1}A^HWA = (A^TWA)^{-1}A^TW.
\]

It is also easily determined that \(WA = A\) (equivalently \(A^TWA = A^T\)) and \(A^TA = I\), so that

\[
A^+ = A^T.
\]
2. $N$ direct, but noisy, ground-based radar measurements are made of the unknown approximately constant speed of an unidentified aircraft entering your airspace. You are to determine the least squares estimate of the speed from the sensor as follows:

(a) *Completely specify appropriate input and output Hilbert spaces, assuming the standard inner-products, and set up an inverse problem of the form $y = Ax$, specifying the elements and dimensions of the matrix $A$.

(b) i) Give the rank of $A$ and the dimensions of the four fundamental subspaces. ii) Is the problem ill-posed? Explain and justify your answers.

(c) i) Construct the adjoint operator of $A$. ii) Construct the pseudoinverse of $A$.

(d) Determine the least-squares estimate of the unknown speed.

(a) Let $s_i \approx x, i = 1, \cdots, N$ denote the noisy measurements of the unknown (scalar) speed $x$. This can be written as

\[
\begin{pmatrix}
  s_1 \\
  \vdots \\
  s_N
\end{pmatrix} = \begin{pmatrix}
  1 \\
  \vdots \\
  1
\end{pmatrix} x \quad \text{or} \quad y = Ax \quad \text{for} \quad y = \begin{pmatrix}
  s_1 \\
  \vdots \\
  s_N
\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}
  1 \\
  \vdots \\
  1
\end{pmatrix}.
\]

We have $y \in \mathcal{Y} = \mathbb{R}^N$ with $\langle y_1, y_2 \rangle = y_1^T y_2$ and $x \in \mathcal{X} = \mathbb{R}^1$ with $\langle x_1, x_2 \rangle = x_1 x_2$.

(b) i) The $N \times 1$ matrix $A$ has one nonzero column so the rank of $A$ is obviously equal to 1, $\text{rank}(A) = r = 1$. Because the inner products are unweighted we have that the adjoint is given by $A^* = A^T$. We have that $\dim \mathcal{R}(A) = \dim \mathcal{R}(A^T) = r = 1$, $\dim \mathcal{N}(A) = 1 - r = 0$, and $\dim \mathcal{N}(A^T) = N - r = N - 1$. ii) Because $A$ has full column rank, the nullspace is trivial, so uniqueness of any solution is guaranteed. However, $r = 1 < N$, so that $A$ is not onto and generically the inverse problem is inconsistent and a true solution does not exist.

(c) i) As noted, because the inner products are unweighted, the adjoint is given by $A^* = A^T$.

ii) Because $A$ has full column rank, The pseudoinverse is given by

\[
A^+ = (A^T A)^{-1} A^T = \frac{1}{N} (1 \cdots 1).
\]

(d) The least squares solution is given by

\[
\hat{x} = A^+ y = \frac{1}{N} \sum_{i=1}^{N} s_i = \text{sample average of the noisy measurements}.
\]
3. Consider the matrix \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) viewed as a mapping of a real Hilbert space \( \mathcal{X} \) into itself, \( A : \mathcal{X} \to \mathcal{X} \). Assume \( \mathcal{X} \) has the standard inner product (i.e., has identity weighting matrix).

(a) Give the rank of \( A \) and the dimensions of the range of \( A \) and the nullspace of \( A \).

(b) Show that \( A \) is self-adjoint and show that as a consequence \( \mathcal{X} = \mathcal{N}(A) \perp \mathcal{R}(A) \).

(c) i) Give a basis for the range of \( A \). ii) Give a basis for the nullspace of \( A \).

(d) Describe in what sense(s) the inverse problem \( y = Ax \) is ill-posed, if at all, for the given matrix \( A \). Justify your answer(s).

(e) Let \( y = (1 - 1) \) and find the minimum-norm least-squares to the problem \( y = Ax \). Justify your answer.

(f) Let \( y = (1) \) and find the minimum-norm least-squares to the problem \( y = Ax \). Justify your answer.

(a) The \( 2 \times 2 \) matrix \( A \) obviously has one linearly independent column (equivalently, row) and therefore has a rank of \( r = 1 \). The dimension of the range of \( A \) is \( r = 1 \) and the dimension of the nullspace is \( 2 - r = 1 \).

(b) Because the inner product weighting matrices are the identity matrix, the adjoint is given by \( A^* = A^T \). However, \( A \) is symmetric, \( A^T = A \), and therefore \( A \) is self-adjoint \( A^* = A \). Thus
\[
\mathcal{X} = \mathcal{N}(A^*) \perp \mathcal{R}(A) = \mathcal{N}(A) \perp \mathcal{R}(A) .
\]

(c) i) A column of \( A \), or any multiple of a column of \( A \), will serve as a basis for the range of \( A \). ii) The vector \( (1) \), or any multiple of this vector, will serve as a basis for the nullspace of \( A \).

(d) \( A \) is not one-to-one (i.e., its nullspace is nontrivial) and therefore solutions to the linear inverse problem \( y = Ax \) lack uniqueness. Furthermore \( A \) is not onto, so generically a solution to the inverse problem will not exist.

(e) For the given value of \( y \) we have \( y \in \mathcal{N}(A) = \mathcal{N}(A^*) = \mathcal{R}(A^*) \perp \mathcal{R}(A) \). Therefore the orthogonal projection of \( y \) onto \( \mathcal{R}(A) \) is given by \( \hat{y} = 0 \). All least squares solutions are therefore given by
\[
0 = \hat{y} = Ax .
\]

Of all such least squares solutions, as a consequence of the projection theorem the one with minimum norm is obviously given by \( \hat{x} = 0 \).

(f) In this case, for the given value of \( y \) we have \( y \in \mathcal{R}(A) \). It is obvious that all solutions must have the form \( x = (\alpha \beta) \) with \( \alpha + \beta = 1 \). However, the least squares solution must also satisfy \( \hat{x} \in \mathcal{R}(A^*) = \mathcal{R}(A) \), so that \( \hat{x} \propto (1) \). The only such solution is given by \( \hat{x} = \frac{1}{2}(1) \).