The Singular Value Decomposition

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Given a complex \( m \times n \) matrix \( A \), we can view it as a linear mapping between the \emph{un–weighted} finite dimensional Hilbert spaces \( \mathcal{X} = \mathbb{C}^n \) and \( \mathcal{Y} = \mathbb{C}^m \), \( A : \mathcal{X} \to \mathcal{Y} \). Note that with these inner products the adjoint of \( A \) is given by its hermitian transpose, \( A^* = A^H \).

Let the rank of \( A \) be denoted by \( r = \text{rank}(A) \), which is the dimension of the range of \( A \), \( \dim(\mathcal{R}(A)) = r \) and the dimension of the range of the adjoint of \( A \), \( \dim(\mathcal{R}(A^H)) = r \). Let the so–called \emph{nullity} \( \nu = n - r \) denote the dimension of the nullspace of \( A \), \( \dim(\mathcal{N}(A)) = \nu \) and let \( \mu = m - r \) denote the dimension of the null space of the adjoint of \( A \), \( \dim(\mathcal{N}(A^H)) = \mu \) (this latter nullspace is also called the left–nullspace of \( A \)).

The SVD is given by the factorization

\[
A = U \Sigma V^H = (U_1 \quad U_2) \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_2^H \end{pmatrix} = U_1 S V_1^H = \sum_{i=1}^{r} \sigma_i u_i v_i^H ,
\]

where \( U \) is \( m \times m \), \( V \) is \( n \times n \), \( \Sigma \) is \( m \times n \), \( S \) is \( r \times r \), \( U_1 \) is \( m \times r \), \( U_2 \) is \( m \times \mu \), \( V_1 \) is \( n \times r \), \( V_2 \) is \( n \times \nu \), \( u_i \) denotes a column of \( U \), and \( v_i \) denotes a column of \( V \). There are precisely \( r \) (nonzero) real singular values of \( A \) rank ordered as

\[
\sigma_1 \geq \cdots \geq \sigma_r > 0 ,
\]
and

\[ S = \text{diag}(\sigma_1 \cdots \sigma_r). \]

The columns of \( U \) form an orthonormal basis for \( \mathcal{Y} = \mathbb{C}^m, \langle u_i, u_j \rangle = u_i^H u_j = \delta_{i,j}, \]
\( i, j = 1, \cdots m. \) Equivalently, \( U \) is a unitary matrix, \( U^H = U^{-1} \). The \( r \) columns of \( U_1 \) form an orthonormal basis for \( \mathcal{R}(A) \) and the \( \mu \) columns of \( U_2 \) form an orthonormal basis for \( \mathcal{N}(A^H) \).

The columns of \( V \) form an orthonormal basis for \( \mathcal{X} = \mathbb{C}^n, \langle v_i, v_j \rangle = v_i^H v_j = \delta_{i,j}, i, j = 1, \cdots n, \) or, equivalently, \( V^H = V^{-1} \). The \( r \) columns of \( V_1 \) form an orthonormal basis for \( \mathcal{R}(A^H) \) and the \( \nu \) columns of \( V_2 \) form an orthonormal basis for \( \mathcal{N}(A) \).

With the SVD at hand the Moore–Penrose pseudoinverse can be determined as

\[ A^+ = V_1 S^{-1} U_1^H = \sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^H. \]

An important check to ascertain that one has correctly constructed the pseudoinverse would be to test that it obeys the four Moore–Penrose pseudoinverse conditions

\[
\begin{align*}
(AA^+)^H &= AA^+; \\
(A^+A)^H &= A^+A; \\
AA^+A &= A; \\
A^+AA^+ &= A^+.
\end{align*}
\]

These conditions are necessary and sufficient for \( A^+ \) to be the pseudoinverse of \( A \).

The various projection operators can be determined as\(^1\)

\[
\begin{align*}
P_{\mathcal{R}(A)} &= U_1 U_1^H = AA^+ = I - P_{\mathcal{N}(A^H)}; \\
P_{\mathcal{N}(A^H)} &= U_2 U_2^H = I - P_{\mathcal{R}(A)}; \\
P_{\mathcal{R}(A^H)} &= V_1 V_1^H = A^+ A = I - P_{\mathcal{N}(A)}; \\
P_{\mathcal{N}(A)} &= V_2 V_2^H = I - P_{\mathcal{R}(A^H)}. 
\end{align*}
\]

Note that because it is possible to construct the projection operators in more than one way, one can test for correctness by doing so and checking for consistency of the answers. This is also another way to determine that the pseudoinverse has been correctly determined.

The columns of \( U \) are the \( m \) eigenvectors of \( (AA^H) \) and are known as the \textit{left singular vectors} of \( A \). Specifically, the columns of \( U_1 \) are the \( r \) eigenvectors of \( (AA^H) \) having associated nonzero eigenvalues \( \sigma_i^2 > 0, i = 1, \cdots r \), while the columns of \( U_2 \) are the remaining \( \mu \) eigenvectors of \( (AA^H) \), which have zero eigenvalues.

The \( n \) columns of \( V \) are the eigenvectors of \( (A^HA) \) and are known as the \textit{right singular vectors} of \( A \). Specifically, the columns of \( V_1 \) are the \( r \) eigenvectors of \( (A^HA) \) having associated nonzero eigenvalues \( \sigma_i^2 > 0, i = 1, \cdots r \), while the \( \nu \) columns of \( V_2 \) are the eigenvectors of \( (A^HA) \), which have zero eigenvalues.

\(^1\)That these are projection operators can be readily shown by testing for the condition of idempotency and self-ajointness.