ECE 174 Midterm — Fall 2010 — Solutions

1. (35 pts) Let the $m \times n$ matrix A represent a linear mapping between two *complex* Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ with inner-product weighting matrices given by Ω and W respectively. (You can assume that all vectors are represented canonically.)

   (a) *Derive* the form of the adjoint of A, from the fundamental definition of the adjoint.

   All of the solutions to Problem 1, except the last part, can be found in the homework solutions. They are straightforward. For example the first solution is:

$$\begin{aligned}
\langle y, Ax \rangle &= y^H W A x \\
&= y^H W A \Omega^{-1} \Omega x \\
&= \left(\Omega^{-1} A^H W y \right)^H \Omega x \\
&= \langle \left(\Omega^{-1} A^H W \right) y, x \rangle \\
&\implies A^* = \Omega^{-1} A^H W
\end{aligned}$$

   (b) Show the relationships between the range spaces and nullspaces of A and its adjoint. (Just show. You do *not* have to derive these relationships.) Draw a simple picture to illustrate these relationships.

   The relationships are:

$$\mathbb{C}^n = \mathcal{X} = \mathcal{N}(A)^\perp \oplus \mathcal{N}(A) = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$$

   and

$$\mathbb{C}^m = \mathcal{Y} = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp = \mathcal{R}(A) \oplus \mathcal{N}(A^*) \, .$$

   You should be able to draw simple pictures that capture these relationships (in particular the orthogonality relationships that exist between the complementary subspaces).

   (c) Consider the inverse problem $y = Ax$. i) *Derive* the algebraic condition for a least-squares solution to exist. ii) *Derive* the algebraic condition for a least-squares solution to be the *minimum norm* least-squares solution.

   See the homework solutions.

(d) *Derive* explicit expressions for the pseudoinverse of A that are possible if it is known that A has full rank. Place them in their simplest possible form. (Hint: There are two different conditions that you have to consider.)

`See the homework solutions.`

(e) Construct the pseudoinverse of A for Ω, W, and A given by

$$\Omega = \begin{pmatrix} 4.80 & 1+2j & 0.01+0.02j \\ 1-2j & 9.76 & 2-j \\ 0.01-0.02j & 2+j & 129.21 \end{pmatrix}, \quad W = \begin{pmatrix} 721.839 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $j = \sqrt{-1}$. *Give the pseudoinverse in its simplest possible form.*

Because A has full column rank, the weighting matrix Ω will <u>not</u> show up in the pseudoinverse

$$A^+ = (A^*A)^{-1}A^* = (A^H W A)^{-1} A^H W = (A^T W A)^{-1} A^T W \,.$$

Further, it is straightforward to show that $WA = A$ (equivalently that $A^T W = A^T$), so that $A^+ = (A^T A)^{-1} A^T$. Finally, it is straightforward to show that $A^T A = I$, yielding the answer $A^+ = A^T$.

2. (40pts) Using radar, m noisy measurements, y_i, $i = 1, \cdots, m$, are made of the unknown constant speed (not velocity) of an unidentified aircraft. Determine the least squares estimate of the speed from the sensor data as follows.

(a) Set up an inverse problem of the form $y = Ax$. To do so, *Completely* specify appropriate input and output Hilbert spaces, showing the definitions of the standard inner-products and clearly specifying the elements and dimensions of the matrix A.

`See below.`

(b) i) Give the rank of A and the dimensions of the four fundamental subspaces. ii) Is the problem ill-posed? Explain and justify your answers.

Letting $x \in \mathcal{X} = \mathcal{X} = \mathbb{R}^1$ denote the unknown (scalar) speed, we have m measurements $y_i \sim x$. Let $y \in \mathcal{Y} = \mathbb{R}^m$ denote the vector of m measurements and the $m \times 1$ matrix A be the (column) matrix whose values are all 1's. Obviously the dimension of the domain $\mathcal{X}$ is 1 and the dimension of the codomain $\mathcal{Y}$ is m. The inverse problem is to solve $y \approx Ax$ for y, A, and x as defined. The rank of A is 1, which is the dimension of the range of A and the dimension of the range of A^T. The nullity (the dimension of the nullspace) is 0, and the dimension of the left-nullspace, $\mathcal{N}(A^T)$, is $m-1$. The inner product on $\mathcal{Y}$ is $\langle y, y' \rangle = y^T y' = y_1 y_1' + \cdots y_m y_m'$ and the inner product on $\mathcal{X}$ is $\langle x, x' \rangle = xx'$. The problem is ill-posed because A is not onto, so that in general the system does not have a true solution. (However A is a full rank matrix since it is one-to-one.) Note that we have assumed that the domain and codomain are cartesian, $\Omega = 1$ and $W = I$.

(c) i) Construct the adjoint operator of A. ii) Construct the pseudoinverse of A.

Because the spaces are real and cartesian, $A^* = A^T = (1, \cdots, 1)$. Because A is one-to-one, we have $A^+ = (A^T A)^{-1} A^T = \frac{1}{m}(1, \cdots, 1)$.

(d) Determine the least-squares estimate of the unknown speed. Write the answer in terms of the *individual* measurements y_i.

$$\hat{x} = A^+ y = \frac{1}{m}(y_1 + \cdots + y_m) = \text{sample average of the measurements.}$$

3. (25pts) In $\mathbb{R}^n$, the general equation of a hyperplane (a translated subspace of dimension $n-1$) is given by

$$\omega_1 x_1 + \cdots + \omega_n x_n - b = \omega^T x - b = 0\,,$$

for $x \in \mathbb{R}^n$. What is the minimum distance of the hyperplane from the origin?

`Solve the problem` $\min_x \|x\|^2$ `subject to` $Ax = y$ `with` $A = \omega^T$ `and` $y = b$ `to obtain the vector` x `from the origin to the hyperplane of shortest length. Since` A `is (obviously) onto, the solution is given by`

$$x = A^+ y = A^T (AA^T)^{-1} y = \omega \cdot \frac{1}{\omega^T \omega} \cdot b = \frac{\omega}{\|\omega\|^2}\, b\,.$$

`The minimum distance to the hyperplane is given by the magnitude of this vector,`

$$\|x\| = \frac{|b|}{\|\omega\|} = \frac{|b|}{\sqrt{\omega_1^2 + \cdots + \omega_n^2}}\,.$$