GLOBAL POSITIONING SYSTEM (GPS) ALGORITHM

Overview

By utilizing measurements of the so-called pseudorange between an object and each of several earth orbiting GPS satellites, the object can be very accurately located in space. A general description of GPS can be found in the article [1], which you should read thoroughly, and in the detailed Web site [2]. Additional technical information can be found in [3] and other articles in [4]. Using your knowledge of the nonlinear least-squares problem you will solve the problem of locating an object given GPS pseudorange data. A self-contained technical development, sufficient for you to perform this task, can be found in the appendix to this assignment and in the class lecture supplement on gradient descent GPS algorithms [5].

Signals are sent from earth orbiting satellites to a receiver located on the surface of the earth. In principle, utilizing timing information and knowledge of the satellite orbit, the true range (the receiver-to-satellite distance) to each satellite can be measured. In practice the range measurement is errorful due to a systematic error caused by an inaccurate receiver clock (to keep the cost of commercial GPS units low) and random errors due to atmospheric distortion, communication channel noise, sensor noise, etc. The errorful range measurement is known as a pseudorange in order to distinguish it from the unknown true range (see the Appendix below). Because of the speed of light; the non-relativistic speeds of the satellites and receiver; and the close proximity of the satellites and object, it is assumed that during the time period needed to collect all the pseudorange information the satellites and receiver are motionless.

Notation and Problem Statement

All locations are given with respect to a fixed-earth (geostationary) reference coordinate system located at the center of the earth. Distances are given in units of Earth Radii (ER) (using an average value of $1\text{ER} = 6,370\text{km}$). Note that according to [1], GPS satellites are located at about $4.14\text{ER}$ from the center of the earth ($20,000\text{km}$ above the earth’s surface).

Use the notation $S_\ell = (x_\ell, y_\ell, z_\ell)^T =$ position of $\ell^{th}$ Satellite, $\ell = 1, \cdots 4$; $S = (x, y, z)^T =$ position of receiver (the Station to be tracked); $R_\ell =$ true range to the receiver located at $S$ from the satellite located at $S_\ell$; and $\Delta S_\ell = S - S_\ell =$ satellite-to-receiver distance vector. For fixed satellite location, $S_\ell$, the true range, $R_\ell$, is a nonlinear function of the receiver location, $S$, and is given by,

$$R_\ell(S) = \|S - S_\ell\| = \|\Delta S_\ell\| = \sqrt{(\Delta S_\ell)^T \Delta S_\ell}, \quad \ell = 1, \cdots , 4.$$  

The pseudorange measurements are denoted by $y_\ell$, and modeled as\footnote{See the attached appendix.}

$$y_\ell = R_\ell(S) + b + \nu_\ell, \quad \ell = 1, \cdots , 4, \quad (1)$$
where the random noise term \( \nu_t \) is i.i.d. with p.d.f. \( N(0, \sigma^2) \). Further discussion of this model is given in the lecture supplement [5]. The (constant) systematic clock bias error \( b \) is caused by an inaccurate clock in the GPS receiver. The number of range measurements taken to each satellite, \( m \), is the same for each satellite.

- Knowing the satellite locations, \( S_\ell, \ell = 1, \cdots, 4 \), and having \( m \) pseudorange measurements, \( y_\ell \), to each of the \( \ell \) satellites, you are to estimate the receiver location, \( S \), and the clock bias, \( b \), using the Gradient Descent and Gauss methods for solving nonlinear least squares problems.

**Simulation Parameter Values.** For the purposes of generating synthetic data, the receiver position, \( S \), and the satellite positions, \( S_\ell \), are taken to be as follows (in \( ER \) units):

\[
S = (1.000000000, 0.000000000, 0.000000000)^T \\
S_1 = (3.585200000, 2.070000000, 0.000000000)^T \\
S_2 = (2.927400000, 2.927400000, 0.000000000)^T \\
S_3 = (2.661200000, 0.000000000, 3.171200000)^T \\
S_4 = (1.415900000, 0.000000000, 3.890400000)^T .
\]

The receiver is assumed to be at mean sea level. The clock bias error, \( b \), is taken to be

\[
b = 2.354788068 \times 10^{-3} ER,
\]

which is equivalent to \( b = 15 \text{ km} \), or approximately 0.05 millisecond.\(^2\) As discussed in the appendix, \( b > 0 \) corresponds to the clock on the receiver running fast compared to the clocks on the satellites. Note the degree of numerical round-off precision required to locate an object to well within a kilometer (\( 1 \text{ km} = 1.569858713 \times 10^{-4} \text{ ER} \)). By default Matlab uses double precision to minimize numerical round-off error, which should be adequate for our problem.

**Procedure**

1. **Linearization.** Assume a nominal receiver location, \( \hat{S} \), and clock bias \( \hat{b} \) and linearize the pseudorange equation (1) about the nominal values. Discuss whether the linearized approximation gets better or worse as the true range increases.

2. **Algorithm Development.** Formulate the *Steepest Descent* and *Gauss’* methods, explicitly showing the actual equations you will use. Identify the step size you will use. Identify your termination criteria. Remember that, in general, it is assumed that \( m \) pseudorange measurements have been taken to each satellite. To accomplish this step, you will need to carefully read and understand the material presented in Lecture Supplements 2 and 3.

\(^2\)Using \( c = 299,792.458 \text{ km/sec} \). The specified value of \( b \), then, is the instantaneous clock accuracy needed to locate an object to within 15 kilometers. Note that to attain the accuracy needed to locate an object to within 10 meters requires that we reduce this clock imprecision by a factor of at least \( 1/1,500 \).
3. **Simulation.** Generate synthetic data and test and compare the Steepest Descent and Gauss’ Methods. For your initial estimate of the vehicle location, \(\hat{S}(0)\), use

\[
\hat{S}(0) = (0.93310, 0.25000, 0.258819)^T ER.
\]

(The initial estimate \(\hat{S}(0)\) corresponds to a location which is about 0.5\(km\) above sea level and about 2,330\(km\) off of the actual location, \(S\). Note that this is a very crude initial guess—as a comparison, the distance between San Diego and Los Angeles is less than 200\(km\).) For your initial estimate of the clock bias use \(\hat{b}(0) = 0\).

You will need to generate pseudorange measurements for three different noise levels: \(\sigma = 0.00000\) (no noise); \(\sigma = 0.00040\) (a range error standard deviation of about 2.5\(km\)); \(\sigma = 0.00400\) (standard deviation of about 25\(km\)). Using this data you should:

(a) Apply the Steepest Descent Algorithm to locate the vehicle for the three different noise cases. For the no-noise case do this for \(m = 1\). For the two nonzero noise cases, do this for \(m = 1, 4, 16, \text{and } 256\). Show the error in your position estimates using units of \textbf{meters}.

(b) Repeat step (a) for the Gauss-Newton method. Again, use units of \textbf{meters} in the error analysis. For the two nonzero noise cases, compare the error (in meters) in your estimates to the predicted (from the linearized dynamics) error standard deviations (\textit{in meters}) given from

\[
\text{Var}\{\hat{X} - X\} = \left(H(\hat{X})^T \Sigma^{-1} H(\hat{X})\right)^{-1},
\]

where \(X = (S^T b)^T \in \mathbb{R}^4\) as described in the lecture supplement [5].

4. **Discussion.** Write up your results using our standard format. Show and discuss the results of Steps 1–3 in some detail, including the observed convergence rates of steepest descent versus the Gauss-Newton method. Present plots showing the loss function \(\ell(k)\), receiver position estimate error \(\|\hat{S}(k) - S\|\text{ in units of meters}\), and clock bias estimate error \(|\hat{b}(k) - b|\) also \textit{in units of meters}, as a function of iteration step, \(k\).

**References**


APPENDIX: The Pseudorange Equation.

Let us focus on a single satellite (say the $\ell$-th one). At time $t_\ell$ the satellite transmits to the receiver its precise location in space, $S_\ell$, and the time, $t_\ell$, of transmission. The clocks on the satellites are very expensive atomic clocks of the highest accuracy. Because of the quality of the atomic clocks on the satellites, we assume that they all agree and are error free.

Because of the finite (albeit extremely high) speed of light, the signal from satellite $\ell$ is received at a slightly later time $t > t_\ell$. The true elapsed time of propagation of the signal is therefore $t - t_\ell$. Denoting the speed of light by $c$, we can determine the range, $R_\ell$, between the satellite and the receiver from

$$ R_\ell = c(t - t_\ell), \tag{2} $$

assuming that $c$ is constant along the communication path and that the path is a straight line. (In fact, atmospheric and other propagation effects can affect this assumption.)

Unfortunately from a modeling perspective (but fortunately for the consumer!), the clock located at the receiver is not an expensive atomic clock, but rather an affordable clock chosen to keep the cost of the receiver down. Thus, in general the receiver believes that it receives the signal at a time $t' \neq t$. The receiver clock synchronization error can be defined as $\Delta t = t' - t$. If $\Delta t = t' - t > 0$, the receiver clock is “fast” (i.e., it “ticks” too fast), while if $\Delta t = t' - t < 0$, the receiver clock is “slow.” The “pseudo-elapsed time” is given by $t' - t_\ell > 0$ (which we assume is positive) and corresponds to the time of transit of the signal as determined by the receiver clock. Thus the receiver calculates the range (erroneously) as

$$ c(t' - t_\ell) = c(t - t_\ell) + c(t' - t) = R_\ell + c(t' - t) = R_\ell + b, \tag{3} $$

where

$$ b = c(t' - t) = c\Delta t \tag{4} $$

is the range error due to the clock bias. Note that $b$ has units of length. Also note that once a good estimate of $b$ is available, we can determine the clock synchronization error in units of time as $\Delta t = c^{-1}b$.

Because of the possibility of unmodeled signal propagation effects and signal processing errors (e.g., round-off errors), instead of directly using equation (3) as our model for the pseudorange we add a random noise component, $\nu_\ell$, to the right hand side of (3) arriving at the pseudorange measurement model

$$ y_\ell = c(t' - t_\ell) = R_\ell + b + \nu_\ell, $$
which was earlier presented in (1). Notice that the information sent from satellite $\ell$ (namely $t_\ell$ and $S_\ell$) enables us to compute the pseudorange $y_\ell = c(t' - t_\ell)$ and construct the functions

$$
\frac{\partial R_\ell(S)}{\partial S} = \frac{\partial}{\partial S} \|S - S_\ell\| = \frac{\partial}{\partial S} \sqrt{(S - S_\ell)^T (S - S_\ell)}
$$

needed to perform the least squares estimation.