ECE 174 Homework # 4, Due Tuesday 5/12/09

New Midterm Date

The midterm is now scheduled for Tuesday, May 19, 2009.

The midterm exam is closed notes and book. No electronic devices (i.e., no cell phones, no calculators, etc.) at all are allowed; bring your own paper and pen (no pencil) only.

There may be some problems on the midterm taken unchanged from the homework and/or sample midterm problems, so you are encouraged to study them well. In addition, there may also be new problems on the exam.

Reading

Meyer §4.6 and §5.13.

If you want to understand the determinant function as a “volume operator” (this is optional but highly recommended) you should carefully study the material presented in Meyer Example 5.13.2 on pages 431–433 (especially the connection to Gram–Schmidt and QR) and Meyer Example 6.1.4 on pages 468–469. An excellent discussion of determinants as volume operators can be found in the textbook Linear Algebra: an Introductory Approach by C.W. Curtis, which can be checked out from the S&E library.

Homework

1. Using the abstract definition of an inner product on a complex hilbert space, prove the following properties. (The complex conjugate of a scalar α is denoted by $\overline{\alpha}$.)

   a) $\langle x_1, \alpha x_2 \rangle = \langle \overline{\alpha} x_1, x_2 \rangle$.
   b) $\langle \alpha_1 x_1 + \alpha_2 x_2, x \rangle = \overline{\alpha_1} \langle x_1, x \rangle + \overline{\alpha_2} \langle x_2, x \rangle$.

2. Let $A : \mathcal{X} \rightarrow \mathcal{Y}$, $B : \mathcal{X} \rightarrow \mathcal{Y}$, and $C : \mathcal{Y} \rightarrow \mathcal{Z}$ be linear mappings as shown between complex finite–dimensional hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. Let $\alpha$ and $\beta$ be complex numbers with complex conjugates $\overline{\alpha}$ and $\overline{\beta}$. As discussed in lecture, the adjoint operator, $A^*$, is defined by

   $\langle A^* x_1, x_2 \rangle = \langle x_1, Ax_2 \rangle$.

Using the abstract definition of an inner product on a complex hilbert space prove the following properties.

   a) $(\alpha A)^* = \overline{\alpha} A^*$.
   b) $(A + B)^* = A^* + B^*$.
   c) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$.
   d) $A^{**} = A$.
   e) $(CA)^* = A^* C^*$.
   f) $A^* \alpha y = \alpha A^* y$.
   g) $A^*(y_1 + y_2) = A^* y_1 + A^* y_2$.
   h) $A^*(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^* y_1 + \alpha_2 A^* y_2$. 


Figure 1: The three resistor values are given and fixed, as is the desired target current $I$. In Problem 4 you are to determine the voltages $V_1$, $V_2$, and $V_3$ which will attained the target current while minimizing the power dissipated in the circuit. You must also determine the optimum (minimum) value of the power dissipated in the circuit.

Note that property (f) shows that the adjoint operator $A^*$, like $A$ itself, is also a linear operator.

3. Let $A : X \to Y$ be an $m \times n$ linear mapping between two Hilbert spaces as shown. Let

$$\langle x_1, x_2 \rangle = x_1^H \Omega x_2 \quad \text{and} \quad \langle y_1, y_2 \rangle = y_1^H W y_2.$$ 

(a) Derive the adjoint operator $A^*$ in terms of $A$, $\Omega$, and $W$.

(b) For $r(A) = n$, derive the pseudoinverse of $A$ and show that it is independent of the weighting matrix $\Omega$.

(c) For $r(A) = m$, derive the pseudoinverse of $A$ and show that it is independent of the weighting matrix $W$.

4. Minimum Power-Dissipation Problem. For the extremely simple circuit shown in Figure 1 you are to find the optimal voltages $V_1$, $V_2$, and $V_3$ which minimize the heat dissipated in the resistors while attaining the specified desired output current $I$ in the following two ways.

(a) Set up and solve a minimum norm linearly–constrained optimization problem using as “state” the vector whose components are the unknown voltages.

(b) Repeat, but now use as “state” the vector whose components are the currents through the resistors, $I_1$, $I_2$, and $I_3$. 
In both cases set up the appropriate mathematical framework by showing: i) the linear mapping $A$; ii) the constraint equation $y = Ax$; iii) the inner products used; iv) the adjoint operator $A^*$; v) the pseudoinverse $A^+$; vi) the derived optimal voltages; and the value of the optimal (minimal) power dissipation in the circuit.

5. Thinking geometrically (i.e., in terms of the four fundamental subspaces) prove Fredholm’s Alternative: Given the system $Ax = b \neq 0$. One, and only one, of the following two statements is true:
   
   a) $Ax = b$ has a solution.
   b) $\langle b, y \rangle \neq 0$ with $A^*y = 0$.

6. Meyer Problem 4.6.7. Set up and solve using the geometric perspective described in the class and in section 5.13 of Meyer. Interpret the variable $x$ as “time” in units of days (say, before and after a “zero day”) and $y$ as a measure quantity to be fit for predictive purposes (say the price per share of some stock you are interested in). Use both of your fits to predict the value of a share of stock on day 6.