Analysis of Multiple Antenna Systems with Finite-Rate Feedback Using High Resolution Quantization Theory

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Abstract

This paper considers the development of a general framework for the analysis of transmit beamforming methods in multiple antenna systems with finite-rate feedback. Inspired by the results of classical high resolution quantization theory, the problem of finite-rate quantized communication system is formulated as a general fixed-rate vector quantization problem with side information available at the encoder (or the quantizer) but unavailable at the decoder. The framework of the quantization problem is sufficiently general to include quantization schemes with general non-mean square distortion functions, and constrained source vectors. The result of the asymptotic distortion analysis of the proposed general quantization problem is presented, which extends the vector version of Bennett's integral. Specifically, tight lower and upper bounds on the average asymptotic distortion are proposed. The proposed general methodology provides a powerful analytical tool to study a wide range of finite-rate feedback systems. To illustrate the utility of the framework, the analysis of a finite-rate feedback MISO beamforming system over i.i.d. Rayleigh flat fading channels is derived. Numerical and simulation results are presented to further confirm the accuracy of the analytical results.

I. INTRODUCTION

Communication systems using multiple antennas have recently received much attention due to their promise of providing significant capacity increases. The performance of the multiple antenna systems depends heavily on the availability of the channel state information (CSI) at the transmitter (CSIT) and at the receiver (CSIR). Most of the MIMO system design and analysis adopt one of two extreme CSIT assumptions, *complete CSIT* and *no CSIT*. In this paper, we consider systems with CSI assumptions in between these extremes. We assume perfect CSIR is available at the receiver, and focus our attention on MIMO systems where CSI is conveyed from the receiver to the transmitter through a finite-rate feedback link. Recently, several interesting papers have appeared, proposing design algorithms as well as analytically quantifying the performance of the finite-rate feedback multiple antenna systems [1]- [7].

Most past works on the analysis of finite-rate feedback MIMO systems have adopted one of three approaches. The first is to approximate the channel quantization region corresponding to each code point based on the channel geometric property. Mukkavilli

This research was supported in part by CoRe grant No.02-10109 sponsored by Ericsson and in part by the U. S. Army Research Office under the Multi-University Research Initiative (MURI) grant # W911NF-04-1-0224.

et. al. [1] derived a universal lower bound on the outage probability of quantized MISO beamforming systems with arbitrary number of transmit antennas t over i.i.d. Rayleigh fading channels. The second approach is based on approximating the statistical distribution of the key random variable that characterizes the system performance. This approach was used by Xia et. al. in [2] [3] and Roh et. al. in [4]- [6], where the authors analyzed the performance of MISO systems over i.i.d. Rayleigh fading channels, and obtained closed form expressions of the capacity loss (or SNR loss) in terms of feedback rate B and antenna size t. The third approach adopted by Narula et. al. in [7] is based on relating the quantization problem to the rate distortion theory, where the authors obtained an approximation of the expected loss of the received SNR due to finite-rate quantization of the beamforming vectors in an MISO system. However, all these approaches are case-specific and quite limited, i.e., i.i.d. and mainly MISO channels, and hard to extend to more complicated schemes.

This paper attempts to provide a general framework for the analysis of quantized feedback multiple antenna systems by exploiting the similarities between classical fixed-rate source coding and the channel quantization. For example, in the fixed-rate quantization problem, the encoder attempts to describe a random source using a finite number of bits with the goal being to minimize a chosen distortion measure (for example, a power of the Euclidean norm of the quantization error). In multiple antenna feedback systems, the channel state information is described using a finite number of bits with the goal being to optimize a given performance metric (such as the received SNR, system mutual information rate or BER). These similarities would be extremely helpful in the design and analysis of finite-rate feedback MIMO systems as they would benefit from the vast body of source coding theory, particularly high resolution quantization theory and VQ-based codebook design methodology. Although several authors have remarked on this similarity (including [1] [3] and [7]), the exact and deeper connection between the two fields still remains elusive. A closer examination reveals that there are enough differences between the problems that a direct use of high resolution results from source coding is not feasible. Fortunately, however, it is possible to extend some of the results to the problem at hand and provide an interesting general framework for analyzing a wide range of finite-rate feedback systems.

Without narrowing the scope to a specific multi-antenna channel quantization scheme, this paper formulates the channel quantization as a general fixed-rate vector quantization problem with attributes tailored to meet the general issues that arise in feedback-based communication systems. These new attributes include side information available at the encoder (or quantizer) but unavailable at the decoder, general non-mean square distortion functions, and source vectors with constraints. Asymptotic distortion analysis of the proposed general quantization problem is provided by extending Bennett's classic analysis [8] as well as its corresponding vector extensions [9] [10]. To be specific, tight lower and upper bounds of the average asymptotic distortion are proposed. The utility of the framework is demonstrated by using it to analyze specific feedback communication systems. Due to space limitations, this paper only provides results for a finite-rate feedback MISO beamforming systems over i.i.d. Rayleigh flat fading channels. Numerical and simulation results are presented which further confirm the tightness of the proposed asymptotic distortion bounds. As indicated earlier, the proposed methodology from the source coding perspective provides a powerful analytical tool to study a wide range of finite-rate feedback systems and is not limited only to this particular case. It can be used to analyze more complicated schemes such as MISO systems over correlated fading channels, MIMO systems (i.i.d. and correlated channels), and even MISO (or MIMO) multicarrier systems over frequency selective fading channels. The established framework is versatile enough to provide analysis of quantizers with mismatched channel statistics and transformed codebooks. These topics are the subject of [11] and [12].

II. Generalized Vector Quantization Framework and Asymptotic Analysis

In this section, the finite-rate feedback-based multiple antenna system is formulated as a generalized fixed-rate vector quantization problem and analyzed by adapting tools from high resolution quantization theory.

A. Motivation for Generalization

To better understand the need for this generalization, an illustrative example is useful. For this purpose, consider a MISO system with t transmit antennas and a single receive antenna where the CSI to be quantized is the vector channel realization $\mathbf{h} \in \mathbb{C}^{\mathsf{T}}$. In contrast to the classical quantization problem, where the encoder and the codebook are designed to minimize the distortion between the source variable and its quantized representation, the design of finite-rate MISO feedback systems is a generalized vector quantization problem because of the following key differences:

1) *Redundant Parameters:* Not all channel parameters need to be quantized. For example, only the channel directional information $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$ is required by the transmitter of a MISO system which employs maximum ratio transmission (MRT). Therefore, it is redundant to directly quantize the channel instantiation $\mathbf{h}$.
2) *Constrained Vector Parameterization:* The actual variable to be quantized may have certain constraints. In the example of quantized MRT beamforming, the vector $\mathbf{v} \in \mathbb{C}^{\mathsf{T}}$ is constrained to be unit-norm and hence lies on the unit hyper-sphere.
3) *Encoder Side Information:* The information which is not the quantization objective, for example the gain $\|\mathbf{h}\|$ of the MISO channel, can be utilized as side information at the encoder to improve the quantization performance.
4) *Non mean-squared performance metric:* The distortion measure, for example effective SNR, system capacity, or bit error rate, is often a more general non-mean-square error function and even parameterized by the side information.

Due to the above-mentioned differences, high resolution quantization theory results from classical source coding cannot be directly applied to the design and analysis of finite-rate feedback systems. In order to take advantage of the vast body of literature on source coding, the analysis must be extended to allow for encoder side information, constrained quantization variables and non-mean-squared distortion measures.

B. Problem Formulation

It is assumed that the source variable $\mathbf{x}$ can be decomposed as $(\mathbf{y}, \mathbf{z})$, where vector $\mathbf{y} \in \mathbb{Q}$ represents the actual quantization variable of dimension k_{q} and $\mathbf{z} \in \mathbb{Z}$ is the additional side information of dimension k_{z}. The *encoder side information* $\mathbf{z}$ is available at the encoder but not at the decoder. Source variable $\mathbf{y}$ and side information $\mathbf{z}$ have joint probability density function given by $p(\mathbf{y}, \mathbf{z})$. This paper considers a fixed-rate (B bits) quantizer with $N = 2^B$ quantization levels. Based on a particular source realization $\mathbf{x}$, the encoder (or the quantizer) represents vector $\mathbf{y}$ by one of the N vectors $\widehat{\mathbf{y}}_1, \widehat{\mathbf{y}}_2, \cdots, \widehat{\mathbf{y}}_N$, which form the codebook. The encoding or the quantization process is denoted as $\widehat{\mathbf{y}} = \mathcal{Q}(\mathbf{y},\, \mathbf{z})$. The distortion of a finite-rate quantizer is defined as

$$D = E_{\mathbf{x}}\Big[D_{\mathrm{Q}}\big(\mathbf{y}, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big)\Big] \;, \tag{1}$$

where $D_{\mathrm{Q}}\big(\mathbf{y}, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big)$ is a general *non mean-squared distortion* function between $\mathbf{y}$ and $\widehat{\mathbf{y}}$ that is parameterized by $\mathbf{z}$. It is further assumed that function D_{Q} has a continuous second order derivative (or Hessian matrix w.r.t. to $\mathbf{y}$) $\mathbf{W}_{\mathbf{z}}(\widehat{\mathbf{y}})$ with the $(i,j)^{\text{th}}$ element given by

$$w_{i,j} = \frac{1}{2}\cdot \frac{\partial^2}{\partial\, y_i \partial\, y_j}\bigg|_{\mathbf{y}=\widehat{\mathbf{y}}} D_{\mathrm{Q}}\big(\,\mathbf{y}\,, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big) \;. \tag{2}$$

Viewed from a conventional source coding perspective, the described general quantization problem is equivalent to the quantization of a mixed density source with each source component having probability density given by $p\big(\mathbf{y}\big|\mathbf{z}\big)$, and general distortion function $D_{\mathrm{Q}}\big(\mathbf{y}, \widehat{\mathbf{y}}\,;\, \mathbf{z}\big)$ parameterized by the encoder side information $\mathbf{z}$, which is also the index information of the source component.

C. Asymptotic Distortion Integral

Under high resolution assumptions, the asymptotic distortion of a finite-rate feedback system can be represented by the following form [13], which is similar to Bennett's integral provided in [8] and its vector extension provided in [9]

$$D = E\Big[D_{\mathrm{Q}}\big(\,\mathbf{y}\,, Q(\mathbf{y}, \mathbf{z})\,;\, \mathbf{z}\big)\Big] = 2^{-\frac{2B}{k_{\mathrm{q}}}} \int_{\mathbb{Z}}\int_{\mathbb{Q}} m\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}_{\mathbf{z}}(\mathbf{y})\big) p\big(\mathbf{y},\, \mathbf{z}\big) \lambda\big(\mathbf{y}\big)^{-\frac{2}{k_{\mathrm{q}}}}\; d\,\mathbf{y}\, d\,\mathbf{z}, \tag{3}$$

where $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ denotes the asymptotic projected Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$ as N approaches infinity. In equation (3), $\lambda(\mathbf{y})$ is a function representing the relative density of the codepoints, which is referred to as the *point density*, such that $\lambda(\mathbf{y})\, d\,\mathbf{y}$ is approximately the fraction of quantization points in a small neighborhood of $\mathbf{y}$. Function $m\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}\big)$ is the *normalized inertial profile* that represents the asymptotic normalized distortion or the relative distortion of the quantizer $\mathcal{Q}$ at position $\mathbf{y}$ conditioned on side information $\mathbf{z}$ with Voronoi shape $\mathbb{E}$. Both $\lambda(\mathbf{y})$ and $m\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}\big)$ are the key performance determining characteristics that can be used to analyze the effects of different system parameters, such as source distribution, distortion function, quantization rate etc., on the finite-rate quantizer.

The normalized inertial profile of an optimal quantizer is defined as the minimum inertia of all admissible regions $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$, i.e.

$$m_{\text{opt}}\big(\mathbf{y}\,;\, \mathbf{z}\big) \stackrel{\triangle}{=} \min_{\mathbb{E}_{\mathbf{z}}(\mathbf{y})\in\mathcal{H}_{\mathrm{Q}}} \; m\big(\mathbf{y}\,;\, \mathbf{z}\,;\, \mathbb{E}_{\mathbf{z}}(\mathbf{y})\big) \;, \tag{4}$$

where $\mathcal{H}_{\mathrm{Q}}$ representing the set of all admissible tessellating polytopes that can tile space $\mathbb{Q}_{\mathbf{z}}$. The optimal inertial profile can be tightly lower bounded (or approximated) by

$$m_{\text{opt}}\big(\mathbf{y}\,;\, \mathbf{z}\big) \gtrapprox \widetilde{m}_{\text{opt}}\big(\mathbf{y}\,;\, \mathbf{z}\big) = \frac{k_{\mathrm{q}}}{k_{\mathrm{q}}+2}\cdot \left(\frac{\big|\mathbf{W}_{\mathbf{z}}(\mathbf{y})\big|}{\kappa_{k_{\mathrm{q}}}^2}\right)^{1/k_{\mathrm{q}}}, \qquad \kappa_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \;. \tag{5}$$

D. Minimization of the Distortion Integral & Different Distortion Bounds

Due to the new attribute of the encoder side information, the generalized vector quantization problem can be viewed as quantizing a multi-component source variable with different distortion functions. Therefore, the codebook should be designed to match the overall distortion averaged over all source components in the sense that both the Voronoi shape and the point density function are optimized. Average system distortion of the optimal quantizer are characterized and bounded in the following.

- *Asymptotic Distortion Lower Bound*

The distortion integral given by equation (3) allows the minimization of the system distortion by optimizing the choice of the Voronoi shape $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ as well as the point density function $\lambda(\mathbf{y})$. Therefore, by substituting the lower bound of the inertial profile (5) derived from the hyper-ellipsoidal approximation into the distortion integral (3) and optimizing $\lambda(\mathbf{y})$ w.r.t. to the overall distortion, the following distortion lower bound of the optimal quantizer can be obtained:

$$D_{\text{Opt}} \geq \widetilde{D}_{\text{Low},1} = \left(\int_{\mathbb{Q}} \left(\widetilde{m}^{\text{w}}_{\text{opt}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} d\,\mathbf{y} \right)^{\frac{2+k_{\text{q}}}{k_{\text{q}}}} \cdot 2^{-\frac{2B}{k_{\text{q}}}} \,, \tag{6}$$

where D_{Opt} represents the distortion of the optimal quantizer, and $\widetilde{m}^{\text{w}}_{\text{opt}}(\mathbf{y})$ is the average optimal inertial profile defined as

$$\widetilde{m}^{\text{w}}_{\text{opt}}(\mathbf{y}) = \int_{\mathbb{Z}} \widetilde{m}_{\text{opt}}(\mathbf{y}\,;\,\mathbf{z}) \cdot p(\mathbf{z}|\,\mathbf{y})\; d\,\mathbf{z}\;. \tag{7}$$

The optimal point density that minimizes the asymptotic system distortion is given by

$$\lambda^{*}(\mathbf{y}) = \left(\widetilde{m}^{\text{w}}_{\text{opt}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \cdot \left(\int_{\mathbb{Q}} \left(\widetilde{m}^{\text{w}}_{\text{opt}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} d\,\mathbf{y} \right)^{-1} . \tag{8}$$

It can be shown that this bound is asymptotically achievable in high dimensions by using a random coding argument.

- *An Alternative Distortion Lower Bound*

If the side information $\mathbf{z}$ is not only available at the encoder but also accessible at the receiver, multiple codebooks $\widehat{\mathbf{y}}_{\mathbf{z},1}, \widehat{\mathbf{y}}_{\mathbf{z},2}, \cdots, \widehat{\mathbf{y}}_{\mathbf{z},N}$ (indexed by the side information $\mathbf{z}$) can be utilized and designed to match each source component. In this case, conditioned on a particular instantiation $\mathbf{z}$, the asymptotic distortion by quantizing a source $\mathbf{y}$ with distribution $p(\mathbf{y}|\,\mathbf{z})$ and distortion function $D_{\text{Q}}(\mathbf{y},\,\widehat{\mathbf{y}}\,;\,\mathbf{z})$ can be lower bounded by [9],

$$D_{\text{opt}}(\mathbf{z}) \geq \widetilde{D}_{\text{Low},2}(\mathbf{z}) = \left(\int_{\mathbb{Q}} \left(\widetilde{m}_{\text{opt}}(\mathbf{y}\,;\,\mathbf{z}) \cdot p(\mathbf{y}|\,\mathbf{z}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} d\,\mathbf{y} \right)^{\frac{2+k_{\text{q}}}{k_{\text{q}}}} \cdot 2^{-\frac{2B}{k_{\text{q}}}} \,. \tag{9}$$

In order to achieve the lower bound $D_{\text{Low},2}(\mathbf{z})$, each component quantizer (parameterized by $\mathbf{z}$) has independent codebook with optimized point density $\lambda^{*}_{\mathbf{z}}(\mathbf{y})$ given by

$$\lambda^{*}_{\mathbf{z}}(\mathbf{y}) = \left(m_{\text{opt}}(\mathbf{y}\,;\,\mathbf{z}) \cdot p(\mathbf{y}|\,\mathbf{z}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \cdot \left(\int_{\mathbb{Q}} \left(m_{\text{opt}}(\mathbf{y}\,;\,\mathbf{z}) \cdot p(\mathbf{y}|\,\mathbf{z}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} d\,\mathbf{y} \right)^{-1} . \tag{10}$$

In the generalized vector quantization problem, due to the unavailability of encoder side information at the decoder, a quantization scheme with multiple codebooks and different point density functions $\lambda_{\mathbf{z}}(\mathbf{y})$ that are matched to each source component, is not feasible. Hence, an alternative distortion lower bound $\widetilde{D}_{\text{Low,2}}$, which is itself a lower bound on $D_{\text{Low,1}}$, can be obtained

$$D_{\text{Opt}} \geq \widetilde{D}_{\text{Low,1}} \geq \widetilde{D}_{\text{Low,2}} = E_{\mathbf{z}}\Big[\widetilde{D}_{\text{Low,2}}(\mathbf{z})\Big] = \int_{\mathbb{Z}} \widetilde{D}_{\text{Low,2}}(\mathbf{z}) \cdot p(\mathbf{z})\; d\,\mathbf{z} \;. \tag{11}$$

- *Asymptotic Distortion Upper Bound*

If the side information $\mathbf{z}$ is unavailable both at the encoder and the decoder, the generalized quantization problem reduces to be a classical fixed-rate vector quantization problem with single component source. In this case, the source variable $\mathbf{y}$ has marginal distribution $p(\mathbf{y})$ and average distortion function D_{Q}^{W} given by

$$D_{\text{Q}}^{\text{w}}(\mathbf{y}, \widehat{\mathbf{y}}) = \int_{\mathbb{Z}} D_{\text{Q}}(\mathbf{y}, \widehat{\mathbf{y}}\,; \mathbf{z}) \cdot p(\mathbf{z}|\, \mathbf{y})\; d\,\mathbf{z} \;, \tag{12}$$

whose corresponding (average) sensitivity matrix can be represented as

$$\mathbf{W}^{\text{w}}(\widehat{\mathbf{y}}) = \int_{\mathbb{Z}} \mathbf{W}_{\mathbf{z}}(\widehat{\mathbf{y}}) \cdot p(\mathbf{z}|\, \widehat{\mathbf{y}})\; d\,\mathbf{z} \;. \tag{13}$$

Due to the fact that the generalized vector quantizer benefits from the availability of the side information at the encoder, the distortion of a side-information-aided quantizer is less than that of quantizing a mixed-component source without side information. Therefore, the average distortion of the side-information-absent quantizer is actually a distortion upper bound of the generalized vector quantizer, which is given by

$$D_{\text{Opt}} \leq \widetilde{D}_{\text{Upp}} = \left(\int_{\mathbb{Q}} \left(\widetilde{m}_{\text{Upp}}^{\text{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} d\,\mathbf{y} \right)^{\frac{2+k_{\text{q}}}{k_{\text{q}}}} \cdot 2^{-\frac{2B}{k_{\text{q}}}} \;, \tag{14}$$

where $\widetilde{m}_{\text{Upp}}^{\text{w}}(\mathbf{y})$ is a tight approximation of the inertial profile obtained by applying the same hyper-ellipsoidal approximation on the cell shape as in equation (5),

$$\widetilde{m}_{\text{Upp}}^{\text{w}}(\mathbf{y}) = \frac{k_{\text{q}}}{k_{\text{q}}+2} \cdot \left(\frac{\left|\mathbf{W}^{\text{w}}(\mathbf{y})\right|}{\kappa_{k_{\text{q}}}^{2}} \right)^{\frac{1}{k_{\text{q}}}} \;. \tag{15}$$

- *Losses Due in the Context of Side Information*

Armed with the above-derived bounds and their corresponding interpretations, the performance loss for quantization with side information can be quantified. First, the loss due to ignorance of the side information at the decoder, where the point density is constrained to be independent of the side information, gives rise to a performance loss:

$$L_{\text{dec}} = \widetilde{D}_{\text{low,1}} / \widetilde{D}_{\text{low,2}} \;. \tag{16}$$

Next, consider the loss due to ignorance of the side information at both the encoder and decoder. In this case, the cell-shapes are constrained to be constant and are optimized under an "averaged" distortion measure. The performance loss in this case is given by:

$$L_{\text{enc}} = \widetilde{D}_{\text{Upp}} / \widetilde{D}_{\text{low,1}} \;. \tag{17}$$

This term represents the additional loss due *solely* to encoder ignorance, and so the total loss of a system with no access to the side information, relative to a system in which both the encoder and receiver have the side information, is given by $L_{\text{tot}} = L_{\text{enc}} \cdot L_{\text{dec}}$. Note that these loss functions specify the penalty in terms of excess distortion, and so the minimum loss is 1. The units can be converted into bits per dimension as $\frac{1}{2}\log_2(L)$.

E. Asymptotic Analysis of Constrained Source

The analysis provided above is for the case that the input source $\mathbf{y}$ is a free random vector of dimension k_{q}. In some situations, it is required to quantize the k_{q} dimensional source vector $\mathbf{y} \in \mathbb{Q}$ subject to a multi-dimensional constraint function $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ of size $k_{\text{c}} \times 1$. In this case, the proposed asymptotic distortion analysis is still valid with the following modifications. First, the degrees of freedom in $\mathbf{y}$ reduce from k_{q} to $k'_{\text{q}} = k_{\text{q}} - k_{\text{c}}$. Second, the sensitivity matrix is replaced by its constrained version $\mathbf{W}_{\text{c},\,\mathbf{z}}(\mathbf{y})$ given by

$$\mathbf{W}_{\text{c},\,\mathbf{z}}(\mathbf{y}) = \mathbf{V}_2^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}(\mathbf{y}) \cdot \mathbf{V}_2 \ , \tag{18}$$

where $\mathbf{V}_2 \in \mathbb{R}^{k_{\text{q}} \times k'_{\text{q}}}$ is an orthonormal matrix with its columns constituting an orthonormal basis for the orthogonal compliment of the range space $\mathcal{R}\big(\frac{\partial}{\partial \mathbf{y}}\, \mathbf{g}(\mathbf{y})\big)$. By utilizing a similar approach as that used in obtaining equation (5), the normalized inertial profile for the constrained source $\mathbf{y}$ can be lower bounded by

$$m_{\text{c-opt}}(\mathbf{y}\,;\,\mathbf{z}) \geq \widetilde{m}_{\text{c, opt}}(\widehat{\mathbf{y}}_i\,;\,\mathbf{z}) = \frac{k'_{\text{q}}}{k'_{\text{q}}+2} \cdot \left(\frac{\left|\mathbf{W}_{\text{c},\,\mathbf{z}}(\widehat{\mathbf{y}}_i)\right|}{\kappa^2_{k'_{\text{q}}}} \right)^{\frac{1}{k'_{\text{q}}}} . \tag{19}$$

Lastly, the multi-dimensional integrations used in evaluating the average distortions are over the constrained space $\mathbf{g}(\mathbf{y}) = 0$. For example, the asymptotic distortion lower bound $D_{\text{Low,1}}$ for constrained source variable, denoted as $\widetilde{D}_{\text{c-Low,1}}$, is given by

$$D_{\text{c-Opt}} \geq \widetilde{D}_{\text{c-Low,1}} = \left(\int_{\mathbf{g}(\mathbf{y})=0} \left(\widetilde{m}^{\text{w}}_{\text{c,opt}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k'_{\text{q}}}{2+k'_{\text{q}}}} d\,\mathbf{y} \right)^{\frac{2+k'_{\text{q}}}{k'_{\text{q}}}} \cdot 2^{-\frac{2B}{k'_{\text{q}}}} \ , \tag{20}$$

where the constrained average inertial profile $\widetilde{m}^{\text{w}}_{\text{c,opt}}(\mathbf{y})$ is given by

$$\widetilde{m}^{\text{w}}_{\text{c,opt}}(\mathbf{y}) = \int_{\mathbb{Z}} \widetilde{m}_{\text{c,opt}}(\mathbf{y}\,;\,\mathbf{z}) \cdot p(\mathbf{z}|\,\mathbf{y})\ d\,\mathbf{z} \ . \tag{21}$$

Following similar derivations, other asymptotic analysis bounds, such as $\widetilde{D}_{\text{c-Low,1}}$, $\widetilde{D}_{\text{c-Low,2}}$, and $\widetilde{D}_{\text{c-Upp}}$ can also be readily obtained.

III. Analysis of Uncorrelated MISO Systems with finite-rate Feedback

Although the analysis of finite-rate quantized MISO beamforming system over i.i.d. Rayleigh fading channels has been investigated in several past works, we revisit this problem from a source coding perspective by formulating it into a general vector quantization problem and provide analysis based on the general framework.

A. System Model

We consider an MISO system with t transmit antennas, one single receive antenna, signaling through a frequency flat block fading channel. The channel impulse response $\mathbf{h}$ is assumed to be perfectly known at the receiver but partially available at the transmitter through CSI feedback. It is assumed that there exists a finite-rate feedback link of B $\left(N = 2^B\right)$ bits per channel update between the transmitter and receiver. To be specific, a codebook $\mathcal{C} = \left\{\widehat{\mathbf{v}}_1, \cdots, \widehat{\mathbf{v}}_N\right\}$, which is composed of unit norm transmit beamforming vectors, is assumed known to both the receiver and the transmitter. Based on the channel realization $\mathbf{h}$, the receiver selects the best code point $\widehat{\mathbf{v}}$ from the codebook and sends the corresponding index back to the transmitter. At the transmitter, vector $\widehat{\mathbf{v}}$ is employed as the transmit beamforming vector, and the system channel model can be represented as

$$y = \mathbf{h}^{\mathsf{H}} \cdot (\widehat{\mathbf{v}} \cdot s) + n = \|\mathbf{h}\| \cdot \langle \mathbf{v}, \widehat{\mathbf{v}} \rangle \cdot s + n \ , \tag{22}$$

where y is the received signal (scalar), $n \sim \mathcal{N}_{\mathrm{c}}(0,1)$ is the additive complex Gaussian noise with zero mean and unit variance, $\mathbf{h}^{\mathsf{H}} \in \mathbb{C}^{1\times t}$ is the MISO channel response with distribution given by $\mathbf{h} \sim \mathcal{N}_{\mathrm{c}}(\mathbf{0}, \mathbf{\Sigma}_{\mathrm{h}})$, and vector $\mathbf{v}$ is the channel directional vector given by $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$. The transmitted signal s is normalized to have a power constraint given by $E\left[s^2\right] = \rho$, with ρ representing the average signal to noise ratio at each receive antenna.

The performance of a finite-rate feedback MISO beamforming system can be characterized by the capacity loss C_{Loss}, which is the expectation of the instantaneous mutual information rate loss $C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}})$ due to the finite-rate quantization of the transmit beamforming vector. This performance metric was also used in [6] and is defined as

$$C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}}) = -\log_2\left(1 - \frac{\rho \cdot \|\mathbf{h}\|^2}{1+\rho\cdot\|\mathbf{h}\|^2} \cdot \left(1 - |\langle \mathbf{v}, \widehat{\mathbf{v}}\rangle|^2\right)\right) \ , \tag{23}$$

B. Problem Formulation

By employing the general framework described in Section II, the finite-rate quantized MISO beamforming system can be formulated as a general fixed rate vector quantization problem. Specifically, the source variable to be quantized is denoted as $\overline{\mathbf{v}} = \left[\mathbf{v}_{\mathrm{R}}^{\mathsf{T}}, \, \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}\right]^{\mathsf{T}}$ of $2t$ real dimensions with $\mathbf{v}_{\mathrm{R}}$ and $\mathbf{v}_{\mathrm{I}}$ representing the real and imaginary parts of the complex channel directional vector $\mathbf{v}$. The encoder side information is denoted as $\alpha = \|\mathbf{h}\|^2$ of dimension $k_\alpha = 1$ representing the power of the vector channel. For vectors in the vicinity of $\widehat{\mathbf{v}}$ (with $\widehat{\mathbf{v}}_{\mathrm{R}}$ and $\widehat{\mathbf{v}}_{\mathrm{I}}$ representing its real and imaginary parts), source variable $\overline{\mathbf{v}}$ is restricted under the constraint function given by

$$\mathbf{g}(\mathbf{v}) = \begin{bmatrix} \mathbf{v}_{\mathrm{R}}^{\mathsf{T}}\mathbf{v}_{\mathrm{R}} + \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}\mathbf{v}_{\mathrm{I}} - 1 \\ \mathbf{v}_{\mathrm{R}}^{\mathsf{T}}\widehat{\mathbf{v}}_{\mathrm{I}} - \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}\widehat{\mathbf{v}}_{\mathrm{R}} \end{bmatrix} = 0 \ , \tag{24}$$

where the first element represents the norm constraint $\|\mathbf{v}\| = 1$, and the second element represents the phase constraint $\measuredangle\langle \mathbf{v}, \widehat{\mathbf{v}}\rangle = 0$. Function $\mathbf{g}(\mathbf{v})$ has size $k_{\mathrm{c}} = 2$, which leads to the actual degrees of freedom of the quantization variable $\mathbf{v}$ to be $k_{\mathrm{q}}' = 2t - 2$. The instantaneous capacity loss due to effects of finite-rate CSI quantization is taken to be the system distortion function $D_{\mathrm{Q}}(\mathbf{v}, \, \widehat{\mathbf{v}}\, ; \, \alpha)$, which is given by the following form according to the definition given by (23)

$$D_{\mathrm{Q}}(\mathbf{v}, \, \widehat{\mathbf{v}}\, ; \, \alpha) = C_{\mathrm{L}}(\mathbf{h}, \, \widehat{\mathbf{v}}) = -\log_2\left(1 - \frac{\rho\alpha}{1+\rho\alpha} \cdot \left(1 - \left|\langle \mathbf{v}, \, \widehat{\mathbf{v}}\rangle\right|^2\right)\right). \tag{25}$$

C. High-Rate Analysis

By utilizing the proposed high-rate distortion analysis, the constrained sensitivity matrix of the finite-rate quantized MISO beamforming system is given by

$$\mathbf{W}_{\mathrm{c},\,\alpha}(\widehat{\mathbf{v}}) = \frac{\rho\alpha}{\ln 2 \cdot (1+\rho\alpha)} \cdot I_{2t-2} \ , \tag{26}$$

Hence, by substituting (26) into the lower bound given by (19), the optimal inertial profile is tightly lower bounded (or approximated) by the following form

$$\widetilde{m}_{\text{c, opt}}\left(\widehat{\mathbf{v}}\,;\,\alpha\right) = \frac{(t-1)\cdot\gamma_t^{-1/(t-1)}\cdot\rho\,\alpha}{\ln 2\cdot t\cdot(1+\rho\,\alpha)}\,, \qquad \gamma_t = \frac{\pi^{t-1}}{(t-1)!}\;. \tag{27}$$

When the elements of the channel response $\mathbf{h}$ are i.i.d. Gaussian distributed, α and $\mathbf{v}$ are statistically independent and the weighted constrained moment of inertia coefficient can be obtained as

$$\widetilde{m}^{\text{w}}_{\text{c, opt}} = \frac{(t-1)\cdot\gamma_t^{-1/(t-1)}}{\ln 2\cdot t}\cdot\left({}_2F_0\left(t+1,\,1;\,;\,-\rho\right)\cdot\rho\right)\;, \tag{28}$$

where ${}_2F_0$ is the generalized hypergeometric function. Therefore, by substituting (28) into the distortion integral (6) and after some manipulations, the average system distortion (or capacity loss) of MISO systems with finite-rate feedback is given by

$$D_{\text{Loss}} = \widetilde{D}_{\text{c-Low,1}} = \frac{(t-1)\,2^{-B/(t-1)}}{\ln 2}\cdot\left({}_2F_0\left(t+1,\,1;\,;\,-\rho\right)\cdot\rho\right)\;, \tag{29}$$

with the optimal point density $\lambda^*(\mathbf{v})$ being a uniform distribution given by

$$\lambda^*(\mathbf{v}) = \gamma_t^{-1}\,, \qquad \mathbf{v}\in\left\{\mathbf{v}\,\middle|\,\mathbf{g}(\mathbf{v})=0\right\}\;. \tag{30}$$

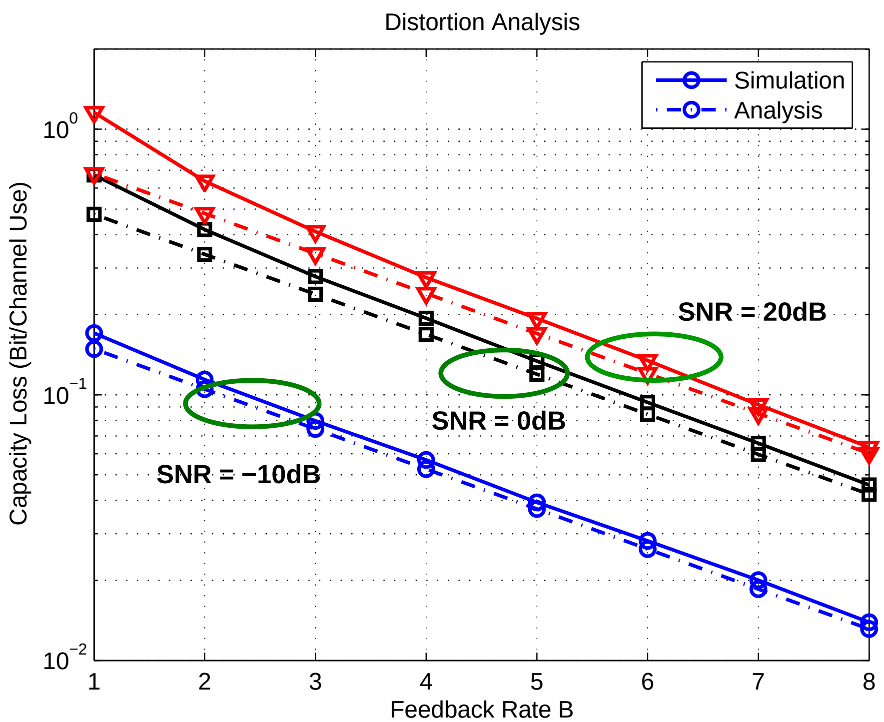

Fig. 1. Capacity loss of a 3×1 MISO transmit beamforming system with finite-rate feedback

Some numerical experiments were conducted to get a better feel for the utility of the bounds. Fig. 1 shows the capacity loss due to the finite-rate quantization of the CSI versus feedback rate B for a 3×1 MISO system over i.i.d. Rayleigh fading channels under different system SNRs at $\rho = -10, 0$ and 20 dB, respectively. The simulation results are obtained from a MISO system using optimal CSI quantizers whose codebooks are generated by the mean squared weighted inner-product (MSwIP) criterion proposed in [5]. The analytical evaluations of the distortion lower bound $D_{\text{c-Low},1}$ provide by equation (29) are also included in the plot for comparisons. It can be observed from the plot that the proposed distortion (or the capacity loss) lower bound is tight and predicts very well the actual system capacity loss obtained from Monte Carlo Simulations.

IV. Conclusion

This paper has developed a general framework for the analysis of multiple antenna systems with finite rate feedback from a source coding perspective. Without narrowing the scope to a specific channel quantization scheme, the problem was formulated as a general fixed-rate vector quantization problem with side information available at the encoder but unavailable at the decoder. The proposed framework is sufficiently general to include quantization schemes with non-mean square distortion functions, and cases where the source vector is constrained. The results of the asymptotic distortion analysis of the proposed general quantization problem was also presented, which extends the vector version of Bennett's integral. More specifically, tight lower and upper bounds of the average asymptotic distortion were provided and related to corresponding classical fixed-rate quantization problems. The proposed general methodology provides a powerful analytical tool to study a wide range of finite-rate feedback systems. To illustrate the utility of the framework, a capacity analysis of the finite-rate feedback MISO beamforming system over i.i.d. Rayleigh flat fading channels was provided. Numerical and simulation results were presented to further confirm the accuracy of the asymptotic distortion bound.

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