Performance Analysis of Coded OFDM Systems Over Frequency-Selective Fading Channels

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***Abstract*— This paper considers the performance analysis of coded OFDM systems over frequency-selective fading channels. It is shown that both the random coding upper bounds and the strong converse lower bounds of a coded OFDM system converge to the channel outage probability for large OFDM block lengths. Thus primary attention is given to the analysis of the outage probability, which is taken as the optimal system performance.**

Instead of evaluating the outage probability numerically, we provide in this paper a simple analytical close form approximation of the outage probability for a coded OFDM system over frequency-selective quasi-static fading channels. Simulation results of the turbo-coded OFDM systems further confirm the approximation of the outage probability.

I. Introduction

ORTHOGONAL frequency-division multiplexing (OFDM) has recently received increased attention due to its capability of supporting high-data-rate communications in frequency-selective fading environments that cause intersymbol interference (ISI) [1][2]. Instead of using a complicated equalizer as in conventional single carrier systems, the ISI in OFDM can be eliminated by adding a guard interval which greatly simplifies the receiver structure. However, in order to achieve the frequency diversity provided by the multipath fading, an appropriate frequency interleaving followed by a forward error correction code is necessary.

Although a considerable amount of research has addressed the design and implementation of coded OFDM systems for frequency-selective fading channels, eg.,[3]-[5], comparatively few of them provide satisfactory performance analysis of such systems because of the complicated nature of this problem. Here we consider a frequency-selective quasi-static fading channel, which is a reasonable assumption for an indoor wireless environment that has multipath fading but exhibits very slow changes over time, modeled as quasi-static. Unlike coding in AWGN channels, where there is one dominant pairwise error probability, related to the minimum distance of a block code or the free distance of a convolutional code, that determines the system performance, all pairwise error probabilities in a fading coded OFDM system decrease as inverse polynomial of the signal-to-noise ratio (SNR). Thus the powerful union-Chernoff bound will be too loose at any range of SNR when the block length is large.

Motivated by the performance analysis results on block fading channels in [10], the random coding upper bounds [6][7] and the strong converse lower bounds [8] of the performance for

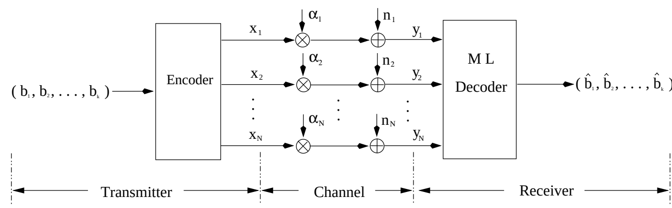

Fig. 1. Frequency model of a coded OFDM system

a coded OFDM system are provided and shown to converge to the channel outage probability for large OFDM block lengths. Hence we focus our primary attention on the channel outage probability and take it as the theoretical achievable performance indicator for the coded OFDM system.

Instead of evaluating the outage probability numerically, a much more simple analytically close form approximation of the outage probability is provided in this paper. Starting from the capacity approximation of the binary input alphabet channel, the outage probability for a coded OFDM system over frequency-selective fading channels is approximated and simplified to a analytically tractable form. Simulation results of the real outage probability as well as the performance of a practical turbo-coded OFDM system further confirms the fitness of this approximation.

The rest of the paper is organized as follows. The system model including the coded OFDM scheme as well as the frequency-selective quasi-static fading channel are described in section II. In section III, the upper and lower bound of the performance for a coded OFDM system is provided and shown to converge to the outage probability for large OFDM block lengths. A simple analytical close form approximation of the outage probability is also provided in this section. Numerical results are given in Section IV. Finally, conclusions are drawn in Section V.

II. System Model

Consider the frequency model of a coded OFDM system illustrated in Fig. 1. A block of k source bits, denoted $\mathbf{b} = (b_1, \cdots, b_k)$, is encoded and mapped into a codeword $\mathbf{x} = (x_1, \cdots, x_N)$. Each x_i is from a complex alphabet $\mathcal{X}$. There are M codewords and the code rate is defined to be $R = (log_2 M)/N$. Note that here we combine encoder, mapper, and interleaver together to form one super encoder. This encoder is followed by N dependent parallel sub-channels, each representing a different subcarrier. According to the tapped-delay-line model [9], the fading coefficients α_i are related to

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the fading envelope c_i through

$$\mathbf{c}^{(t)} = \left[c_1^{(t)}, \cdots, c_L^{(t)}, 0, \cdots, 0\right]^T \in \mathcal{C}^{N\times 1}, \quad (1)$$
$$\boldsymbol{\alpha}^{(t)} = \left[\alpha_1^{(t)}, \cdots, \alpha_N^{(t)}\right]^T \in \mathcal{C}^{N\times 1}, \quad (2)$$
$$\boldsymbol{\alpha}^{(t)} = W_{N\times N} \cdot \mathbf{c}^{(t)} , \quad (3)$$

where the Fourier transformation matrix $W_{N\times N}$ is given by

$$W = \begin{pmatrix} w^0 & w^0 & \dots & w^0 \\ w^0 & w^1 & \dots & w^{N-1} \\ w^0 & w^2 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots \\ w^0 & w^{N-1} & \dots & w^{(N-1)(N-1)} \end{pmatrix}, \; w = e^{-j\frac{2\pi}{N}}. \quad (4)$$

A quasi-static fading environment is adopted in which the fading envelope $\mathbf{c}$ is assumed to be constant during one OFDM block, but independent from block to block. A Rayleigh fading distribution is considered in this paper where the probability density function (pdf) of $|c_i|$ is given by

$$f_{|c_i|}(x) = 2Lx \cdot \exp(-Lx^2), \qquad x \geq 0 \; . \quad (5)$$

It is further assumed that each tap has the same average power (this model can be generalized to have a non-rectangular power profile).

The received output vectors $\mathbf{y}$ are given by

$$y_i = \alpha_i x_i + n_i, \qquad i = 1, \cdots, N \, , \quad (6)$$

where the additive complex Gaussian noise $\{n_i\}$ are $i.i.d$ with variance N_0. The receiver is assumed to have perfect knowledge of the channel state information(CSI) and performs maximum likelihood sequence detection based on the observation of the received vector $\mathbf{y}$.

III. Performance Analysis

A. Upper and Lower Bounds

Malkamaki in [10] introduces a performance analysis technique implementing the random coding upper bound and the strong converse lower bound on quasi-static block fading channels, which results in an excellent convergence of these two bounds as well as the outage probability for large channel block lengths. Motivated by the fact that an OFDM system can also be viewed as an N-parallel block fading channel where each subchannel contains only one single block component, the same method can thus be applied to quasi-static fading OFDM systems.

Implementing the techniques introduced in [10], via slight modifications, we obtain the random coding upper bound (see, e.g., [6], and [7]) and the strong converse lower bound (see, e.g., [7], and [8]) for the OFDM system introduced in Section II.

$$\overline{P_e(\boldsymbol{\alpha})} \leq \begin{cases} 2^{-NE_N(R/\boldsymbol{\alpha})}, & \text{for } 0 \leq R < C_N(\boldsymbol{\alpha}) \; , \\ 1, & \text{for } R \geq C_N(\boldsymbol{\alpha}) \; , \end{cases} \quad (7)$$

$$P_c(\boldsymbol{\alpha}) \leq \begin{cases} 2^{-NE_N^{SC}(R/\boldsymbol{\alpha})}, & \text{for } R \geq C_N(\boldsymbol{\alpha}) \; , \\ 1, & \text{for } 0 \leq R < C_N(\boldsymbol{\alpha}) \; , \end{cases} \quad (8)$$

where

$$E_N(R/\boldsymbol{\alpha}) = \max_{0\leq\rho\leq 1} \Big[\max_q \frac{1}{N} \sum_{i=1}^{N} E_0(\rho, q_i/\alpha_i) - \rho R \Big], \quad (9)$$

$$E_N^{SC}(R/\boldsymbol{\alpha}) = \max_{-1\leq\rho\leq 0} \Big[\min_q \frac{1}{N} \sum_{i=1}^{N} E_0(\rho, q_i/\alpha_i) - \rho R \Big], \quad (10)$$

$$E_0(\rho, q_i/\alpha_i) = -log_2 \left\{ \sum_{y_i} \Big[\sum_{x_i} q_i(x_i) p(y_i/x_i, \alpha_i)^{1/(1+\rho)} \Big]^{1+\rho} \right\} , \quad (11)$$

and

$$C_N(\boldsymbol{\alpha}) = \frac{1}{N} \max_q I(\mathbf{x^N}; \mathbf{y^N}/\boldsymbol{\alpha}) = \frac{1}{N} \sum_{i=1}^{N} \max_{q_i} I(x_i; y_i/\alpha_i). \quad (12)$$

The above formulas describe the upper and lower bounds of the performance of a coded OFDM system with rate $R = (\log_2 M)/N$. Equation (7) is the upper bound of the block error probability averaged over the ensemble codebook conditioned upon the fading coefficients $\boldsymbol{\alpha}$ assuming ML decoding with perfect channel state information. Equation (8) is the upper bound of the correct decoding probability for any code.

Note that in the above bounds, it is very difficult to optimize the distribution $q_i(x)$ under a generalized complex alphabet $\mathcal{X}$. However it is easy to prove that the uniform distribution is optimum when the alphabet is symmetric on the complex plane such as with M-ary PSK. Furthermore, most of the time, we are only interested in an input alphabet that has an equal prior probability distribution. Thus in the following discussions, the input alphabet is restricted to a uniform input distribution. In this paper, the transitional probability $p(y_i/x_i, \alpha_i)$ in (11) is given by

$$p(y_i/x_i, \alpha_i) = \frac{1}{\pi N_0} \exp(-\frac{|y_i - \alpha_i x_i|^2}{N_0}) \; . \quad (13)$$

Finally, it might be insightful to make a comparison between our quasi-static fading OFDM channel and the block fading channel in [10]. First, instead of having independent fading on different sub-channels in a block fading channel, the fading coefficients α_i of different OFDM sub-carriers are correlated. Second, even though each OFDM sub-carrier has block length 1 when viewed as a parallel block fading channel, the convergence of the upper bound and lower bound still exists for OFDM system having large number of subcarriers as is shown in the following section. This is in contrast to the convergence condition of a block fading channel (large block length).

B. Outage Probability and System Performance

Averaging the conditional upper bound (7) and the lower bound (8) over the fading vector $\boldsymbol{\alpha}$ yields

$$\overline{P_e} = E_{\boldsymbol{\alpha}} \Big[\overline{P_e(\boldsymbol{\alpha})}\Big] \leq \int_{\overline{\mathcal{U}}} \Big(2^{-NE_N(R/\boldsymbol{\alpha})}\Big) \cdot f(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \; + \int_{\mathcal{U}} 1 \cdot f(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \, , \quad (14)$$

and

$$P_e = E_{\boldsymbol{\alpha}}\Big[1 - P_c(\boldsymbol{\alpha})\Big]$$
$$\geq \int_{\mathcal{U}} \Big(1 - 2^{-NE_N^{SC}(R/\boldsymbol{\alpha})}\Big) \cdot f(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} + \int_{\overline{\mathcal{U}}} 0 \cdot f(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}, \quad (15)$$

where

$$\mathcal{U} = \{\boldsymbol{\alpha} | C_N(\boldsymbol{\alpha}) \leq R\} \ . \quad (16)$$

When the number of OFDM sub-carriers N is sufficient large, it is obvious to demonstrate that both the upper bound (14) and the lower bound (15) converge to the outage probability $Pr(out)$, which is defined as

$$\Pr(out) = \int_{\mathcal{U}} 1 \cdot f(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = Pr(C_N(\boldsymbol{\alpha}) \leq R) \ . \quad (17)$$

Since the upper bound is the ensemble average performance of all codebooks, but the lower bound is for any code, there must exists at least one coding scheme whose performance is bounded by (14), (15) and converges to (17) as N goes to infinite. From now on, we will focus our attention on the outage probability, which will be used as an indicator of the system performance.

C. *Conditional Sub-channel Capacity*

For simplicity, the input alphabet $\mathcal{X}$ is constrained to be binary, while extension to larger input alphabets is straightforward. According to the channel transition probability (13), the conditional sub-channel capacity is given by

$$C(\gamma_i) = \max_{q_i} I(x_i; y_i/\alpha_i) = \frac{1}{\log 2}\Bigg(2\gamma_i - \int_{y_i} \frac{1}{4\pi\gamma_i} \log\Big(\cosh\big(\Re[y_i]\big)\Big) \cdot \exp\big(-\frac{|y_i - 2\gamma_i|^2}{4\gamma_i}\big)\, dy_i\Bigg), \quad (18)$$

where

$$\gamma_i = \frac{|\alpha_i|^2 \cdot E_s}{N_0} \ . \quad (19)$$

With the capacity expressed in such a complicated form, it is almost impossible to perform any analysis on the outage probability. Fortunately, this binary input conditional channel capacity is well approximated by the following simple analytical form

$$C(\gamma_i) \approx 1 - \exp(-m\gamma_i), \quad m = 1.24 \ . \quad (20)$$

The details of the derivation of this approximation are provided in Appendix A of [12].

Results of the the exact sub-channel capacity given by (18) through numerical integration versus the approximation given by (20) are depicted in Fig. 2. From the plot, it is seen that the exact conditional channel capacity is well approximated and almost identical to the approximated form.

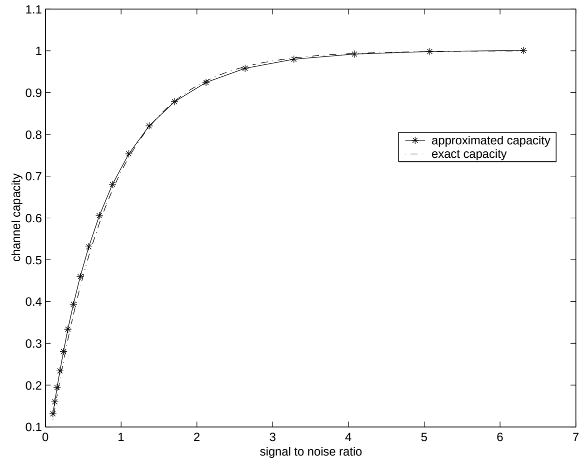

Fig. 2. Approximation of the conditional sub-channel capacity for binary input channels

D. *Approximation of the Outage Probability*

When the channel is frequency selective and spread over L taps, it is quite clear from (18) and (12) that $C_N(\boldsymbol{\alpha})$ is a highly non-linear function of the vector $\boldsymbol{\alpha}$. Generally, it is very difficult or impossible to get the cdf (or pdf) of a random vector's non-linear transformation. In this paper, instead of evaluating the system performance numerically, we provide an approximate but simple analytical form for this outage probability.

Plugging (20) into (12), we have

$$C_N(\boldsymbol{\alpha}) \approx 1 - \frac{1}{N}\sum_{i=1}^{N} \exp(-m\gamma_i), \quad m = 1.24 \ . \quad (21)$$

where γ_i is given by (19). If we perform a Taylor series expansion on each exponential term in the above equation, (21) can be written as

$$C_N(\boldsymbol{\alpha}) \approx 1 - \sum_{j=0}^{\infty}\sum_{i=1}^{N} \frac{(-m\gamma_i)^j}{N \cdot j!} . \quad (22)$$

Using the following fact of an exponential random variable,

$$E\big[\gamma_i^j\big] = j! \cdot E\big[\gamma_i\big]^j \ , \quad (23)$$

it is reasonable to extend this property from the ensemble mean to the sample mean by

$$\frac{1}{N}\sum_{i=1}^{N} \gamma_i^j \approx j! \cdot \Big(\frac{1}{N}\sum_{i=1}^{N} \gamma_i\Big)^j \ . \quad (24)$$

Substituting (24) into (22), the instantaneous channel capacity $C_N(\boldsymbol{\alpha})$ can be further simplified to be

$$C_N(\boldsymbol{\alpha}) \approx 1 - \sum_{j=0}^{\infty} \Big(-m \cdot \frac{1}{N}\sum_{i=1}^{N} \gamma_i\Big)^j = 1 - \frac{1}{1 + m \cdot \frac{1}{N}\sum_{i=1}^{N} \gamma_i} \ . \quad (25)$$

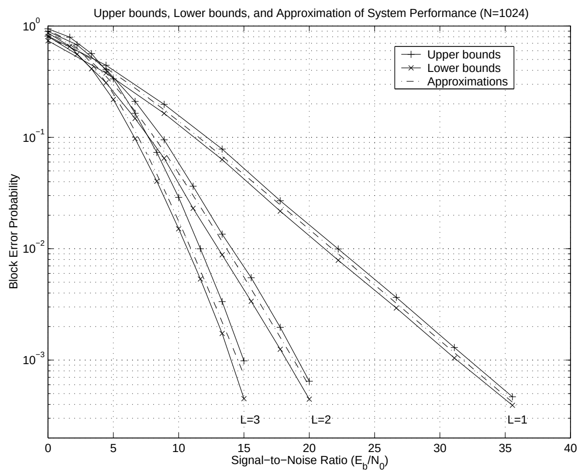

Fig. 3. Random coding upper bound and the strong converse lower bound as well as the approximated outage probability for a coded OFDM system with $N = 1024$ subcarriers over a frequency-selective fading channel with $L = 1, 2, 3$ paths.

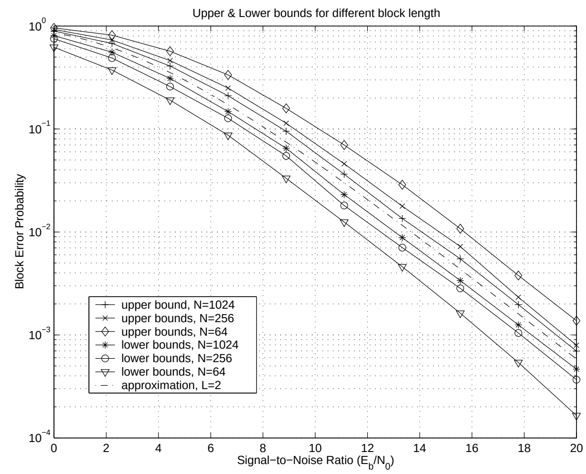

Fig. 4. Upper and lower bounds for three coded OFDM systems of different numbers of subcarriers, $N = 1024, 256, 64$, transmitted over frequency-selective fading channels with $L = 2$ paths.

Plugging (25) into the outage probability definition (17), we can get a much more simplified outage probability form

$$Pr(out) \approx Pr\Big(\frac{1}{N}\sum_{i=1}^{N}\gamma_i \le \frac{R}{(1-R)\cdot m}\Big)\,. \tag{26}$$

According to (3) and (19), we know that the sample mean of γ_i is a quadratic transformation of $\boldsymbol{\alpha}$(or $\mathbf{c}$) given by

$$z = \frac{1}{N}\sum_{i=1}^{N}\gamma_i = \frac{1}{N}\big(\mathbf{c}^H W^H W\,\mathbf{c}\big)\frac{E_s}{N_0} = \big(\mathbf{c}^H\cdot\mathbf{c}\big)\frac{E_s}{N_0}\,. \tag{27}$$

From (5), we know that the c_i are L independent Gaussian random variables. This means that z is a central χ^2 random variable with $2L$ degree of freedom, and the outage probability is given by the cdf of this random variable

$$Pr(out) \approx 1 - \exp\Big(-\frac{RL}{m(1-R)\overline{\gamma_s}}\Big)\sum_{k=0}^{L-1}\frac{1}{k!}\Big(\frac{RL}{m(1-R)\overline{\gamma_s}}\Big)^k, \tag{28}$$

where

$$\overline{\gamma_s} = \frac{E_s}{N_o}\,. \tag{29}$$

When $\overline{\gamma_s} \gg 1$, the outage probability reduces to

$$Pr(out) \approx \frac{1}{L!}\cdot\Big(\frac{RL}{m(1-R)\overline{\gamma_s}}\Big)^L\,. \tag{30}$$

It is quite clear from (30) that the system can achieve maximum diversity order of L, which is exactly the number of paths. However, in order to achieve this maximum diversity, a powerful channel coding scheme should be implemented.

IV. Numerical and Simulation Results

A. Upper Bounds and Lower Bounds

The random coding upper bound averaged over the fading coefficients based upon (7) and (14) as well as the strong converse lower bound (8) and (15) were computed numerically. We assume a binary symmetric channel model (perform hard decisions before ML detection) and a Rayleigh distribution of c_i with a rectangular multipath power profile. Results for these two bounds and the approximated outage probability are depicted in Fig. 3 for a coded OFDM system with $N = 1024$ subcarriers transmitting over a frequency-selective fading channel with several different numbers of paths.

Theoretically, the outage probability is greater than the lower bound but less than the upper bound which is exactly the case shown in the above plot, although only an approximated result is used here. Further from Fig. 3, we see that the lower bound is only about 1dB from the upper bound, which indicates that both of these bounds and also the approximated outage probability are quite tight and a valid performance indication of the system for large block lengths.

To see the sensitivity of the tightness of the bounds in terms of the block length, we depict in Fig. 4 the bounds for the same rate $R = 1/2$ system under three different block lengths. From it, we see that the two bounds and the approximated outage probability are reasonably tight when $N \ge 256$.

B. Outage Probability

Fig. 5 shows the exact outage probability as well as our approximated results for the coded OFDM system with $N = 1024$ subcarriers for various numbers of paths, $L = 2 \sim 9$. From that plot, we see that the exact outage probability is well approximated by (28), especially when the number of independent paths is large. This is expected because the approximation in (24) becomes more accurate when there are a large number of independent random variables.

C. Results of Practical Codes

In order to get an indication of how close we can get to the theoretical outage probability with practical codes, simulations were carried out using various coding schemes. In Fig. 6, and Fig. 7 the simulated block error probability of a rate $1/2$ terminated convolutional code with constraint length $K = 9$ and

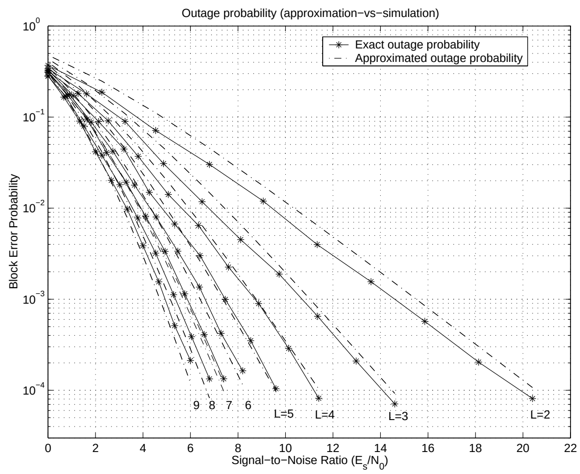

Fig. 5. Exact outage probability vs. the approximated outage probability for a coded OFDM system with $N = 1024$ subcarriers over a frequency-selective fading channel with $L = 2 \sim 9$ paths.

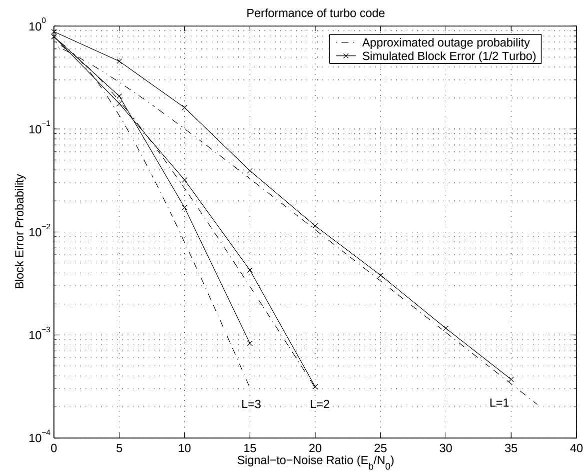

Fig. 7. Block error probability of a rate $1/2$ turbo code with component generator polynomial $(7, 5)$ compared with the approximated outage probability.

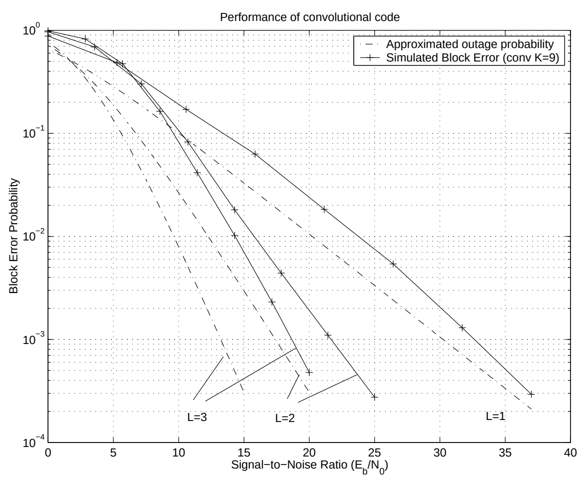

Fig. 6. Block error probability of a rate $1/2$ terminated convolutional code with constraint length $K = 9$ and generator polynomial $(561, 753)$ compared with the approximated outage probability.

generator polynomial $(561, 753)$ as well as a rate $1/2$ turbo code with generator polynomial $(7, 5)$ are compared with the approximated outage probability for different numbers of paths $L = 1, 2, 3$. From the plots above and other simulation results (which are not presented here due to length constraints), we can see that the optimum system performance can be achieved by some near Shannon capacity coding schemes, such as turbo code. This is a direct result of the information theoretical analysis approaches adopted in this paper for the evaluation of the coded OFDM system performance.

V. Conclusion

In this paper, performance analysis of coded OFDM systems are investigated over frequency-selective fading channels. Both the random coding upper bounds and the strong converse lower bounds are derived and shown to converge to the channel outage probability for large OFDM block lengths. Hence the outage probability draws primary attention in this paper and is taken as the optimal performance indicator of a coded OFDM system. Instead of evaluating the outage probability numerically, an approximate but analytically close expression of the outage probability is provided. Numerical results of the exact outage probability as well as the simulation results of a practical turbo-coded OFDM system well demonstrate and further confirms the fitness of this approximation.

Throughout the discussion in the paper, we find this approximation of the outage probability not only provides us a guidance on evaluating various coding schemes for the coded OFDM system, but also serves as a handy tool to compare with other communication systems operating in the same multipath fading environment.

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