Cramer-Rao Lower Bound for Constrained Complex Parameters

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***Abstract*— An expression for the Cramer-Rao lower bound (CRB) on the covariance of unbiased estimators of a constrained complex parameter vector is derived. The application and usefulness of the result is demonstrated through its use in the context of a semi-blind channel estimation problem.**

***Index Terms*— Cramer-Rao Bound, CRB, Constrained Parameters, Channel Estimation, Semi-Blind, MIMO.**

I. Introduction

The CRB serves as an important tool in the performance evaluation of estimators which arise frequently in the fields of communications and signal processing. Most problems involving the CRB are formulated in terms of unconstrained real parameters [1]. Two useful developments of the CRB theory have been presented in later research. The first being a CRB formulation for unconstrained complex parameters given in [2]. This treatment has valuable applications in studying the base-band performance of modern communication systems where the problem of estimating complex parameters arises frequently. A second result is the development of the CRB theory for constrained real parameters [3], [4], [5]. However, in applications such as semi-blind channel estimation one is faced with the estimation of constrained complex parameters. Though one can reduce the problem to that of estimating constrained real parameters by considering the real and imaginary components of the complex parameter vector, the complicated resulting expressions result in loss of insight. Using the calculus of complex derivatives as is often done in signal processing applications, considerable insight and simplicity can be achieved by working with the complex vector parameter as a single entity [1], [6], [7]. We thus present an extension of the result in [3], [4], [5] inspired by the theory in [2] for the case of constrained complex parameters. To conclude, we illustrate its usefulness by an example of a semi-blind channel estimation problem.

II. CRB For Complex Parameters With Constraints

Consider the complex parameter vector $\bar{\gamma} \in \mathbb{C}^{n\times 1}$. Let $\bar{\gamma} \triangleq \bar{\alpha} + j\bar{\beta}$ such that the real and imaginary parameter vectors $\bar{\alpha}, \bar{\beta} \in \mathbb{R}^{n\times 1}$ and $\bar{\xi} \triangleq \left[\bar{\alpha}^T, \bar{\beta}^T\right]^T$. Assume that the likelihood function of the (possibly complex) observation vector $\bar{\omega} \in \Omega$ parameterized by $\bar{\xi}$ is $s(\bar{\omega}; \bar{\xi})$. Let $\hat{\bar{\xi}} : \Omega \rightarrow \mathbb{R}^{2n\times 1}$ be given as $\hat{\bar{\xi}} \triangleq \left[\hat{\bar{\alpha}}^T, \hat{\bar{\beta}}^T\right]^T$, where $\hat{\bar{\alpha}}, \hat{\bar{\beta}}$ are unbiased estimators of $\bar{\alpha}, \bar{\beta}$ respectively. In the foregoing analysis, we define the gradient $\dfrac{dr(\bar{\alpha})}{d\bar{\alpha}} \in \mathbb{R}^{1\times n}$ of a scalar function $r(\bar{\alpha})$ as a row vector:

$$\frac{dr(\bar{\alpha})}{d\bar{\alpha}} \triangleq \left[\begin{array}{cccc} \dfrac{dr(\bar{\alpha})}{d\alpha_1}, & \dfrac{dr(\bar{\alpha})}{d\alpha_2}, & \ldots, & \dfrac{dr(\bar{\alpha})}{d\alpha_n} \end{array} \right]. \tag{1}$$

Let $\bar{\theta} \in \mathbb{C}^{2n\times 1}$ be defined as in [2] by

$$\bar{\theta} \triangleq \left[\begin{array}{c} \bar{\gamma} \\ \bar{\gamma}^* \end{array} \right]. \tag{2}$$

Suppose now that the l complex constraints on $\bar{\theta}$ are given as

$$\mathbf{h}\left(\bar{\theta}\right) = \mathbf{0}, \tag{3}$$

i.e. $\mathbf{h}\left(\bar{\theta}\right) \in \mathbb{C}^{l\times 1}$. We then construct an extended constraint set (of possibly redundant constraints) $\mathbf{f}\left(\bar{\theta}\right) \in \mathbb{C}^{2l\times 1}$ as

$$\mathbf{f}\left(\bar{\theta}\right) \triangleq \left[\begin{array}{c} \mathbf{h}\left(\bar{\theta}\right) \\ \mathbf{h}^*\left(\bar{\theta}\right) \end{array} \right] = \mathbf{0}. \tag{4}$$

An important observation from (4) above is that symmetric complex constraints on these parameters are treated as disjoint. For instance, given the orthogonality of complex parameter vectors $\bar{\theta}_1, \bar{\theta}_2$, i.e. $\bar{\theta}_1^H \bar{\theta}_2 = 0$, the symmetric constraint $\bar{\theta}_2^H \bar{\theta}_1 = 0$ is to be treated as an additional complex constraint and hence $\mathbf{f}(\bar{\theta}) = \left[\bar{\theta}_1^H \bar{\theta}_2, \bar{\theta}_2^H \bar{\theta}_1\right]^T$. The extension of the constraints is akin to the extension of the parameter set from $\bar{\gamma}$ to $\bar{\theta} = [\bar{\gamma}, \bar{\gamma}^*]$ called for when dealing with complex parameters, and the need will become evident from the proof of lemma(1). Reparameterizing $\mathbf{h}\left(\bar{\theta}\right) = \mathbf{h}_R\left(\bar{\theta}\right) + j\mathbf{h}_I\left(\bar{\theta}\right)$ in terms of $\bar{\xi}$, let the set of $2l$ parameter constraints for $\bar{\xi}$ be given by $\mathbf{g}\left(\bar{\xi}\right) = \left.\left[\mathbf{h}_R\left(\bar{\theta}\right)^T, \mathbf{h}_I\left(\bar{\theta}\right)^T\right]^T\right|_{\bar{\theta}=\bar{\alpha}+j\bar{\beta}}$. Employing notation defined in [3] and borrowing the notion of a complex derivative from [1], [6], we define $F\left(\bar{\theta}\right) \in \mathbb{C}^{2l\times 2n}$ as

$$F\left(\bar{\theta}\right) \triangleq \frac{\partial \mathbf{f}(\bar{\theta})}{\partial \bar{\theta}} = \left[\begin{array}{cc} \dfrac{\partial \mathbf{f}\left(\bar{\theta}\right)}{\partial \bar{\gamma}}, & \dfrac{\partial \mathbf{f}\left(\bar{\theta}\right)}{\partial \bar{\gamma}^*} \end{array} \right], \tag{5}$$

It then follows from the properties of the complex derivative [6] that

$$F\left(\bar{\theta}\right) = \frac{1}{2} T \frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}} S, \tag{6}$$

where $T \in \mathbb{C}^{2l\times 2n}, S \in \mathbb{C}^{2n\times 2n}$ are given as

$$T \triangleq \left[\begin{array}{cc} 1 & j \\ 1 & -j \end{array} \right] \otimes \mathbf{I}_{l\times l} \quad , \quad S \triangleq \left[\begin{array}{cc} 1 & 1 \\ -j & j \end{array} \right] \otimes \mathbf{I}_{n\times n}. \tag{7}$$

The non-minimality of the set of complex constraint does not affect the CRB . Alternatively, a minimal set of complex

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constraints can be obtained by first formulating $\mathbf{g}\left(\bar{\xi}\right)$ and then reparameterizing in terms of $\bar{\theta}$. However, such a process involves a tedious procedure of separating the real and imaginary parts, when it might be more natural to consider the complex parameters themselves as in the above example of orthogonality of parameter vectors. Let $rank\left(F\left(\bar{\theta}\right)\right) = k < 2n$. Hence there exists a $U \in \mathbb{C}^{2n\times 2n-k}$ such that U forms an orthonormal basis for the nullspace of $F(\bar{\theta})$ i.e. $F(\bar{\theta})U = \mathbf{0}$. Let the likelihood of the observed data $p(\bar{\omega};\bar{\theta})$ be reparameterized as $s\left(\bar{\omega};\bar{\xi}\right)$ by substituting $\bar{\gamma} = \bar{\alpha} + j\bar{\beta}$, $\bar{\gamma}^* = \bar{\alpha} - j\bar{\beta}$. Define Δ as

$$\Delta \triangleq \frac{\partial \ln p(\bar{\omega};\bar{\theta})}{\partial \bar{\theta}} = \left[\left(\frac{1}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\alpha}} - \frac{j}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\beta}}\right),\right.$$
$$\left.\left(\frac{1}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\alpha}} + \frac{j}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\beta}}\right)\right]^T, \tag{8}$$

where the last equation follows from the definition of $p(\bar{\omega};\bar{\theta})$. Let $J = \mathrm{E}\left\{\Delta^*\Delta^T\right\}$ denote the Fisher information matrix (FIM) for the unconstrained estimation of $\bar{\theta}$. Also assume that

A.1: The parameter vector $\bar{\xi} \in \mathbb{R}^{2n\times 1}$ and the likelihood function $s\left(\bar{\omega};\bar{\xi}\right)$ satisfy the regularity conditions as in [3], [8]. We present them below for the sake of completeness.

(i) $\bar{\xi} \in \Xi$, where $\Xi \subseteq \mathbb{R}^{2n}$.

(ii) $\frac{\partial s\left(\bar{\omega};\bar{\xi}\right)}{\partial \xi_i}$, $i \in \{1, 2, \ldots, 2n\}$ exists and is a.s. finite for every $\bar{\xi} \in \Xi$.

(iii) $\int \left|\frac{\partial^k s\left(\bar{\omega};\bar{\xi}\right)}{\partial \xi_i^k}\right| < \infty$, for every $\bar{\xi} \in \Xi$, and $k = 1, 2$.

(iv) $\mathrm{E}\left\{\left|\frac{\partial s\left(\bar{\omega};\bar{\xi}\right)}{\partial \xi_i}\right|^2\right\} < \infty$, for every $\bar{\xi} \in \Xi$.

We now present a result for the constrained complex estimator $\hat{\bar{\theta}}$ analogous to the real case.

Lemma 1: Under assumption A.1 and constraints given by (3), the constrained estimator $\hat{\bar{\theta}}\colon \Omega \to \mathbb{C}^{n\times 1}$ defined as

$$\hat{\bar{\theta}} \triangleq \begin{bmatrix} \hat{\bar{\alpha}} + j\hat{\bar{\beta}} \\ \hat{\bar{\alpha}} - j\hat{\bar{\beta}} \end{bmatrix} \tag{9}$$

satisfies the property

$$\mathrm{E}\left\{\left(\hat{\bar{\theta}} - \bar{\theta}\right)\Delta^T\right\} UU^H = UU^H. \tag{10}$$

Proof: From the results for constrained real parameter vector in [3], [5] we have

$$\mathrm{E}\left\{\left(\hat{\bar{\xi}} - \bar{\xi}\right)\tilde{\Delta}^T\right\}\tilde{U}\tilde{U}^T = \tilde{U}\tilde{U}^T, \tag{11}$$

where $\tilde{\Delta} = \begin{bmatrix} \frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\alpha}} & \frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\beta}} \end{bmatrix}$, and $\tilde{U} \in \mathbb{C}^{2n\times 2n-k}$ is a basis for the nullspace of $\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}$. Let $\tilde{U} = \left[U_I^T, U_R^T\right]^T$, $U_I, U_R \in \mathbb{R}^{n\times 2n-k}$, $\tilde{\bar{\alpha}} \triangleq \hat{\bar{\alpha}} - \bar{\alpha}$ and $\tilde{\bar{\beta}} \triangleq \hat{\bar{\beta}} - \bar{\beta}$. Then rewriting the above expression in terms of block partitioned matrices we have,

$$\int_\Omega \left(\begin{bmatrix} \tilde{\bar{\alpha}} \\ \tilde{\bar{\beta}} \end{bmatrix}\begin{bmatrix} \frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\alpha}} & \frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\beta}} \end{bmatrix}\begin{bmatrix} U_I \\ U_R \end{bmatrix}\right.$$
$$\left.\times \begin{bmatrix} U_I^T & U_R^T \end{bmatrix} d\bar{\omega}\right) = \begin{bmatrix} U_I \\ U_R \end{bmatrix}\begin{bmatrix} U_I^T & U_R^T \end{bmatrix}. \tag{12}$$

Let $U \in \mathbb{C}^{2n\times 2n-k}$ is defined as

$$U \triangleq \frac{1}{\sqrt{2}}\begin{bmatrix} U_I + j\,U_R \\ U_I - j\,U_R \end{bmatrix}.$$

With some manipulation, (12) can be written in terms of complex matrices as

$$\int_\Omega \left(\begin{bmatrix} \tilde{\bar{\alpha}} + j\tilde{\bar{\beta}} \\ \tilde{\bar{\alpha}} - j\tilde{\bar{\beta}} \end{bmatrix}\left[\frac{1}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\alpha}} - \frac{j}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\beta}},\right.\right.$$
$$\left.\left.\frac{1}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\alpha}} + \frac{j}{2}\frac{\partial \ln s\left(\bar{\omega};\bar{\xi}\right)}{\partial \bar{\beta}}\right] UU^H\, d\bar{\omega}\right) = UU^H,$$

Using (8) and (9), the above equation can be expressed in the form given by (10). It remains to show that U forms a basis for the nullspace of $F\left(\bar{\theta}\right)$. It follows from the definition of $\tilde{U}$ that $\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}\tilde{U} = \mathbf{0}$ and this equality is true if and only if,

$$\frac{1}{\sqrt{2}}\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}\left(\frac{1}{2}SS^H\right)\tilde{U} = \mathbf{0} \tag{13}$$
$$\Leftrightarrow \frac{1}{2\sqrt{2}}T\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}SS^H\tilde{U} = \mathbf{0} \tag{14}$$
$$\Leftrightarrow F\left(\bar{\theta}\right)\left(\frac{1}{\sqrt{2}}S^H\tilde{U}\right) = \mathbf{0}, \tag{15}$$

where the equalities in (13), (14) follow from the facts $\frac{1}{2}SS^H = \mathbf{I}$ and T is invertible, respectively. The matrices S, T have been defined in (7). It can be seen that $U = \frac{1}{\sqrt{2}}S^H\tilde{U}$ and therefore $U \perp F\left(\bar{\theta}\right)$. Moreover, $U^H U = \frac{1}{2}\tilde{U}^T SS^H\tilde{U} = \mathbf{I}_{k\times k}$. Hence U contains orthonormal columns. Showing that it spans the nullspace of $F\left(\bar{\theta}\right)$ completes the proof. Let U not span the nullspace of $F\left(\bar{\theta}\right)$. Then there exists $\mathbf{u} \triangleq \left[\mathbf{u}_a^T, \mathbf{u}_b^T\right]^T$ where $\mathbf{u}_a, \mathbf{u}_b \in \mathbb{C}^{n\times 1}$ such that $F\left(\bar{\theta}\right)\mathbf{u} = \mathbf{0}$ and $U^H\mathbf{u} = \mathbf{0}$. Hence we have $T\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}S\mathbf{u} = \mathbf{0} \Rightarrow \frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}S\mathbf{u} = \mathbf{0}$ as T is an invertible matrix. Let $\tilde{\mathbf{u}} \triangleq S\mathbf{u} = [u_a^T + u_b^T, ju_b^T - ju_a^T]^T$. Since $\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}$ is real we have $\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}\tilde{\mathbf{u}}_R = 0$ where $\tilde{\mathbf{u}}_R$ is the real part of $\tilde{\mathbf{u}}$. Also, it can be observed that $U^H\mathbf{u} = 0 \Rightarrow \tilde{U}^T\tilde{\mathbf{u}} = 0$ and since $\tilde{U}$ is a real matrix, $\tilde{U}^T\tilde{\mathbf{u}}_R = 0$. Thus there exists a real vector viz. $\mathbf{v} \triangleq \tilde{\mathbf{u}}_R \in \mathbb{R}^{2n\times 1}$ such that $\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}\mathbf{v} = \tilde{U}^T\mathbf{v} = 0$ contradicting the assumption that $\tilde{U}$ is a basis for the nullspace of $\frac{\partial \mathbf{g}\left(\bar{\xi}\right)}{\partial \bar{\xi}}$. This completes the proof. ■

Theorem 1: Under assumption A.1 and constraints given by (3), the CRB for estimation of the constrained parameter $\bar{\theta} \in \mathbb{C}^{2n\times 1}$ is then given as

$$\mathrm{E}\left\{\left(\hat{\bar{\theta}} - \bar{\theta}\right)\left(\hat{\bar{\theta}} - \bar{\theta}\right)^H\right\} \geq U\left(U^H J U\right)^{-1}U^H. \tag{16}$$

Proof: Let $P_U = UU^H$ be the projection matrix onto the column space of U and let $W \in \mathbb{C}^{2n\times 2n}$ be an arbitrary matrix. Let $\tilde{\bar{\theta}} \triangleq \left(\hat{\bar{\theta}} - \bar{\theta}\right)$. As in [3] we now consider $\mathrm{E}\left\{\left(\tilde{\bar{\theta}} - WP_U\Delta^*\right)\left(\tilde{\bar{\theta}} - WP_U\Delta^*\right)^H\right\}$. Following a procedure similar to that for real vectors provided in [3], the proof of (16) then follows by making the obvious modifications for complex matrices (i.e. replacing the transpose operator with the hermitian, etc.). ∎

III. A Constrained Matrix Estimation Example

A. Problem Formulation

We consider in this section the problem of pilot assisted semi-blind estimation of a complex MIMO (Multi-Input Multi-Output) channel matrix $H \in \mathbb{C}^{t\times t}$ (i.e. # transmit antennas = # receive antennas = t). Let a total of L pilot symbols be transmitted. The channel input-output relation is represented as

$$\mathbf{y}_k = H\mathbf{x}_k + \mathbf{v}_k \quad , \quad k = 1, 2, \ldots, L, \tag{17}$$

where $\mathbf{y}_k, \mathbf{x}_k \in \mathbb{C}^{t\times 1}$ are the received and transmitted signal vectors at the k-th time instant. $\mathbf{v}_k \in \mathbb{C}^{t\times 1}$ is spatio-temporally uncorrelated Gaussian noise such that $\mathrm{E}\left\{\mathbf{v}_k\mathbf{v}_k^H\right\} = \sigma_n^2\mathbf{I}$. H can be factorized using its singular value decomposition (SVD) as $H = P\Sigma Q^H$ where $P, R \in \mathbb{C}^{t\times t}$ are orthogonal matrices such that $P^H P = Q^H Q = \mathbf{I}$, $\Sigma = diag\left(\sigma_1, \sigma_2, \ldots, \sigma_t\right)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_t > 0$. P, Σ can be estimated using blind techniques. We then employ the pilot data exclusively to estimate the constrained orthogonal matrix Q. More about the significance of such a problem can be found in [9].

B. Cramer-Rao Bound

Let $\tilde{\mathbf{y}}_k = P^H\mathbf{y}_k$,$\tilde{\mathbf{v}}_k = P^H\mathbf{v}_k$. Denote by $\mathbf{q}_i$ the i-th column of the matrix Q. The unconstrained input-output relation for each $\mathbf{q}_i$ can be written as

$$\tilde{\mathbf{y}}_{k,i} = \sigma_i\mathbf{x}_k^H\mathbf{q}_i + \tilde{\mathbf{v}}_{k,i}, \tag{18}$$

where $\mathbf{y}_{k,i}$ denotes the i-th element of $\mathbf{y}_k$ and analogously for $\mathbf{v}_{k,i}$. Define the desired parameter vector to be estimated $\bar{\theta} \triangleq \left[vec\left(Q\right), vec\left(Q^*\right)\right]^T$. It can now be seen that $\bar{\theta}$ is a constrained parameter vector and the constraints are given as

$$\mathbf{q}_i^H\mathbf{q}_i = 1, \quad 1 \leq i \leq t \tag{19}$$

$$\mathbf{q}_i^H\mathbf{q}_j = 0, \quad 1 \leq i < j \leq t. \tag{20}$$

Hence, the set of $t + \binom{t}{2}$complex constraints $\mathbf{h}\left(\bar{\theta}\right)$ is given as $\mathbf{h}\left(\bar{\theta}\right) = \left[\mathbf{q}_1^H\mathbf{q}_1 - 1, \mathbf{q}_1^H\mathbf{q}_2, \mathbf{q}_3^H\mathbf{q}_1, \ldots, \mathbf{q}_t^H\mathbf{q}_t - 1\right]^T$. The extended constraint set $\mathbf{f}\left(\bar{\theta}\right)$ is then given as $\mathbf{f}\left(\bar{\theta}\right) = \left[\mathbf{q}_1^H\mathbf{q}_1 - 1, \mathbf{q}_1^H\mathbf{q}_2, \mathbf{q}_1^H\mathbf{q}_3, \ldots, \mathbf{q}_t^H\mathbf{q}_t - 1, \ldots, \mathbf{q}_1^H\mathbf{q}_1 - 1, \mathbf{q}_2^H\mathbf{q}_1, \mathbf{q}_3^H\mathbf{q}_1, \ldots, \mathbf{q}_t^H\mathbf{q}_t - 1\right]^T$. $\mathbf{f}\left(\bar{\theta}\right)$ can then be employed to compute U. However, it can be noticed that the repeated constraint $\mathbf{q}_i^H\mathbf{q}_i - 1$ for $i = 1, 2, \ldots, t$ is trivially redundant. Eliminating this redundancy, the minimal set of $t + 2\binom{t}{2} = t^2$ set of non-redundant constraints $\tilde{\mathbf{f}}\left(\bar{\theta}\right)$ can be obtained as $\tilde{\mathbf{f}}\left(\bar{\theta}\right) = \left[\mathbf{q}_1^H\mathbf{q}_1 - 1, \mathbf{q}_1^H\mathbf{q}_2, \mathbf{q}_2^H\mathbf{q}_1, \mathbf{q}_1^H\mathbf{q}_3, \mathbf{q}_3^H\mathbf{q}_1, \ldots, \mathbf{q}_t^H\mathbf{q}_t - 1\right]^T$.

$F\left(\bar{\theta}\right)$ is constructed as given in (5), by differentiating $\tilde{\mathbf{f}}\left(\bar{\theta}\right)$ with respect to the parameter vector $\bar{\theta}$. For example, the derivative of constraint # 2 i.e. $\mathbf{q}_1^H\mathbf{q}_2$ is given as $\frac{\partial \mathbf{q}_1^H\mathbf{q}_2}{\partial\theta} = \left[0, \mathbf{q}_1^H, 0, \ldots, \mathbf{q}_2^T, 0, 0, \ldots\right]$, where we have used the fact that $\frac{\partial \mathbf{q}_1^H}{\partial \mathbf{q}_1} = \frac{\partial \mathbf{q}_2}{\partial \mathbf{q}_2^H} = 0$. This result follows from the properties of the complex derivative in [1]. Similarly, $\frac{\partial \mathbf{q}_1^H\mathbf{q}_1}{\partial\theta} = \left[\mathbf{q}_1^H, 0, 0, \ldots, \mathbf{q}_1^T, 0, 0, \ldots\right]$, and so on. The matrix U is an orthogonal basis for the nullspace of $F\left(\bar{\theta}\right)$. Hence, for this example, the matrices $F\left(\bar{\theta}\right) \in \mathbb{C}^{t^2\times 2t^2}$, $U \in \mathbb{C}^{2t^2\times t^2}$ can be written explicitly and are given as

$$\mathbf{F}\left(\bar{\theta}\right) = \begin{bmatrix} \mathbf{q}_1^H & 0 & 0 & \ldots & \mathbf{q}_1^T & 0 & 0 & \ldots \\ 0 & \mathbf{q}_1^H & 0 & \ldots & \mathbf{q}_2^T & 0 & 0 & \ldots \\ \mathbf{q}_2^H & 0 & 0 & \ldots & 0 & \mathbf{q}_1^T & 0 & \ldots \\ 0 & \mathbf{q}_2^H & 0 & \ldots & 0 & \mathbf{q}_2^T & 0 & \ldots \\ \mathbf{q}_3^H & 0 & 0 & \ldots & 0 & 0 & \mathbf{q}_1^T & \ldots \\ 0 & 0 & \mathbf{q}_1 & \ldots & \mathbf{q}_3^T & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{q}_1 & 0 & \mathbf{q}_2 & 0 & \mathbf{q}_3 & \ldots \\ 0 & \mathbf{q}_1 & 0 & \mathbf{q}_2 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ -\mathbf{q}_1^* & -\mathbf{q}_2^* & 0 & 0 & 0 & \ldots \\ 0 & 0 & -\mathbf{q}_1^* & \mathbf{q}_2^* & 0 & \ldots \\ 0 & 0 & 0 & 0 & -\mathbf{q}_1^* & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The simplistic and insightful nature of the above matrices $F\left(\bar{\theta}\right), U$ in terms of the orthogonal parameter vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_t$, is particularly appealing and illustrates the efficacy of using the complex CRB. From Eq(18) and using the results for least-squares estimation [1] the Fisher information matrix $J\left(\bar{\theta}\right) \in \mathbb{C}^{2t^2\times 2t^2}$ for the unconstrained case is given by the block diagonal matrix $J\left(\bar{\theta}\right) = \frac{1}{\sigma_n^2}\left(\mathbf{I}_{2\times 2} \otimes \Sigma^2 \otimes X_pX_p^H\right)$. The complex constrained CRB for the parameter vector $\bar{\theta}$ is then obtained by substituting these matrices in (16).

C. ML Estimate and Simulation Results

We now compute the Maximum-Likelihood (ML) estimate and compare its performance with that predicted by the CRB. The received symbol vectors can be stacked as $\tilde{Y}_p \triangleq \left(\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2, \ldots, \tilde{\mathbf{y}}_L\right)$. Let X_p be defined analogously by stacking the transmitted symbol vectors. Then $\hat{Q}$ the ML estimate of Q is given as a solution of the cost

$$\hat{Q} = \arg\min\left\|\tilde{Y_p}^H - X_p^H Q\Sigma\right\|^2 \quad \text{subject to} \quad QQ^H = \mathbf{I}$$

where the norm $\|\cdot\|$ is the matrix Frobenius norm such that $\|A\|^2 = \mathrm{tr}\left(AA^H\right)$. From [10] the constrained estimate $\hat{Q}$ employing an orthonormal pilot sequence X_p (i.e. $X_pX_p^H = \mathbf{I}$) is given as

$$\hat{Q} = P_pR_p^H \quad \text{where} \quad P_p\Sigma_pR_p^H = \mathrm{SVD}\left(X_p\tilde{Y_p}^H\Sigma\right) \tag{21}$$

Our simulation set-up consists of a 4×4 MIMO channel H (i.e. $t = 4$). A single realization of H was generated as a

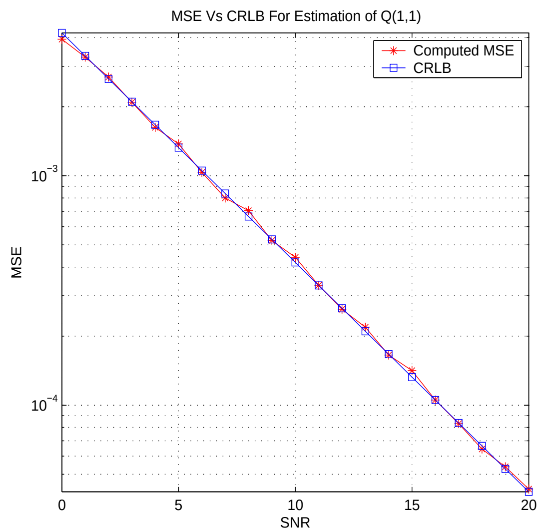

Fig. 1. Computed MSE Vs SNR, $\left|Q(1,1)-\hat{Q}(1,1)\right|^2$

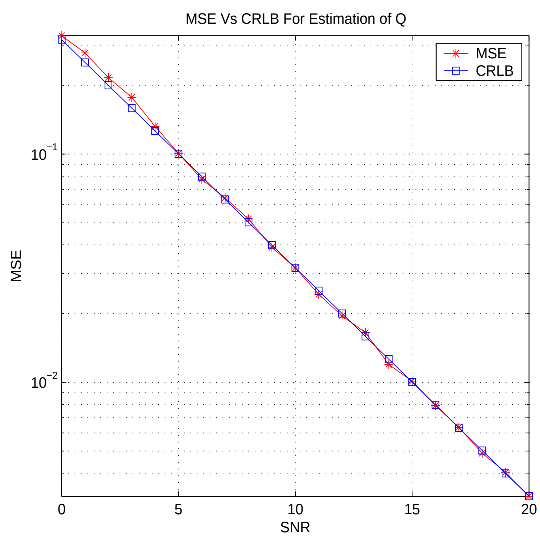

Fig. 2. Computed MSE Vs SNR, $\left\|Q-\hat{Q}\right\|^2$

matrix of zero-mean circularly symmetric complex Gaussian random entries such that the variance of the real and imaginary parts was unity. The source symbol vectors $\mathbf{x} \in \mathbb{C}^{4\times 1}$ are assumed to be drawn from a BPSK constellation and the orthonormality condition is achieved by using the Hadamard structure. The transmitted pilot was assumed to be of length $L = 12$ symbols. The error was then averaged for a fixed H over several instantiations ($N_i = 1000$) of the channel noise $\mathbf{v}_k$. Figure(1) shows the MSE in the 1st element $\hat{Q}(1,1)$ $\left(\text{i.e. } \left|Q(1,1)-\hat{Q}(1,1)\right|^2\right)$ vs its CRB. Similar results were obtained for the CRB of other elements of Q. Figure(2) then shows the total MSE in estimation of Q $\left(\text{i.e. } \left\|Q-\hat{Q}\right\|^2\right)$ vs the trace of the CRB matrix. The ML estimate $\hat{Q}$ can be seen to achieve a performance close to the CRB and its performance progressively improves with increasing SNR.

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