Solutions to Homework 1

Problem 1:

X_1 and X_2 are full column rank $N \times M$ matrices. Let $X_1 = \left[x_1^{(1)}, \cdot, x_M^{(1)}\right]$ and $X_2 = \left[x_1^{(2)}, \cdot, x_M^{(2)}\right]$. Since $\mathcal{R}(X_1) = \mathcal{R}(X_2)$ then $x_1^{(2)}, \cdots, x_M^{(2)}$, the columns of X_2 are in $\mathcal{R}(X_1)$, i.e. $x_l^{(2)} \in \mathcal{R}(X_1), l = 1, \cdots, M$.

Hence,

$$x_l^{(2)} = X_1 t_l, l = 1, \cdots, M$$

or

$$X_2 = \left[x_1^{(2)}, \cdots, x_M^{(2)}\right] = [X_1 t_1, X_1 t_2, \cdots, X_1 t_M] = X_1 [t_1, t_2, \cdots, t_M] = X_1 T,$$

where T is a $M \times M$ square matrices. Since X_1 and X_2 are full column rank, T is non-singular and invertible. Note that T singular would automatically contradict the assumption that X_2 is full column rank and hence has no nullspace.

$$\begin{aligned} X_2 X_2^+ &= X_2 (X_2^H X_2)^{-1} X_2^H = X_1 T (T^H X_1^H X_1 T)^{-1} T^H X_1^H \\ &= X_1 T T^{-1} (X_1^H X)^{-1} (T^H)^{-1} T^H X_1^H = X_1 (X_1^H X_1)^{-1} X_1^H = X_1 X_1^+ \end{aligned}$$

Problem 3: Let $P = P_M P_S$, where P_M and P_S are orthogonal projection operators onto subspaces of dimension M and S respectively. They satisfy $P_M = P_M^H = P_M^2$, and $P_S = P_S^H = P_S^2$. To show P is an orthogonal projection matrix, we have to show that it satisfies $P = P^H = P^2$. We first check the symmetry property.

$$P^H = (P_M P_S)^H = P_S^H P_M^H = P_S P_M.$$

In general, matrix product is not a commutative operation and so $P_M P_S \neq P_S P_M$ and hence

$P \neq P^H$. An easy way to confirm this is with an example. Consider

$$P_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } P_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

It is easy to verify that the above matrices are orthogonal projection matrices.

$$P = P_M P_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \text{ and } P_S P_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

P is clearly not symmetric and $P_M P_S \neq P_S P_M$. So for P to be an orthogonal projection matrix we need

$$P_M P_S = P_S P_M. \tag{1}$$

Next let's check the idempotent property.

$$P^2 = P_M P_S P_M P_S.$$

If $P_M P_S = P_S P_M$, then

$$P^2 = P_M P_S P_M P_S = P_M P_S P_S P_M = P_M P_S P_M = P_M P_M P_S = P_M P_S = P.$$

So property (1) is also sufficient to establish the idempotent property. In summary $P = P_M P_S$ is an orthogonal projection matrix iff $P_M P_S = P_S P_M$.

Now we discuss when this is possible. We will use the notation V_M to denote the subspaces of dimension M, and $V_M^\perp$ its orthogonal complement, i.e. $V_M \oplus V_M^\perp = C^N$. $\oplus$ implies direct sum, that is any vector in C^N can be expressed uniquely as a sum of two vectors, one from V_M and the other from $V_M^\perp$. Similar comments apply to subspace V_S of dimension S. The intersection of the subspaces V_M and V_S is also a subspace denoted by $V_{M\cap S}$. Note that if $P_M P_S$ is an orthogonal projection operator, it will project onto the space $V_{M\cap S}$. Now we can split V_M into two subspaces as shown below

$$V_M = V_{M\cap S} \oplus \tilde{V}_M$$

where $\tilde{V}_M \perp V_{M\cap S}$, and describes the space that is in V_M and not shared with V_S. Hence $C^N = V_M \oplus V_M^{\perp} = V_{M\cap S} \oplus \tilde{V}_M \oplus V_M^{\perp}$, and any vector in C^N can be decomposed into three components

$$x = x_{M\cap S} + \tilde{x}_M + x_M^{\perp}.$$

The relationship between the components and the spaces should be clear from the notation. Then

$$P_S P_M x = x_{M\cap S} + P_S \tilde{x}_M. \tag{2}$$

Similarly $C^N = V_{M\cap S} \oplus \tilde{V}_S \oplus V_S^{\perp}$, and $x = x_{M\cap S} + \tilde{x}_S + x_S^{\perp}$. Then

$$P_M P_S x = x_{M\cap S} + P_M \tilde{x}_S. \tag{3}$$

Since $P_M P_S = P_S P_M$ is an orthogonal projection operator onto $V_{M\cap S}$, it follows that

$$P_M \tilde{x}_S = P_S \tilde{x}_M = 0, \ \ \forall \tilde{x}_S \in \tilde{V}_S \text{ and } \tilde{x}_M \in \tilde{V}_M.$$

This implies $\tilde{V}_S \perp V_M$ and $\tilde{V}_M \perp V_S$ or $\tilde{V}_M \perp \tilde{V}_S$.

In summary,

$$V_M = V_{M\cap S} \oplus \tilde{V}_M, V_S = V_{M\cap S} \oplus \tilde{V}_S, \text{ and } \tilde{V}_M \perp \tilde{V}_S \Leftrightarrow P_M P_S = P_S P_M$$