Filter Bank Applications

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An Important Tool and Resource Easily Accessed in the DSP Domain

> **Sample Rate**
> + By embedding Sample Rate Changes in the filter we can form Linear Time Varying (LTV) or Periodically Time Varying (PTV) filters.
> + In an LTV or PTV filter, we can apply an input signal at one frequency and the response can occur at another frequency.
> + In the sampled data world we call this *Aliasing* (traveling under a different name).
> + **Aliasing is your friend!**

> **As an Aside**
> + Most of us have limited experience with LTV systems.
> + Actually we all share one common LTV system experience, a swing!
> + Every child on the playground masters the **two** modes of swing excitation.
Mode I. Pumping A Swing From a Sitting Position

Rotate Body Mass in Positive Direction when Angular Velocity is Zero and Angular Acceleration is Positive

Additive Growth Per Cycle

Rotate Body Mass in Negative Direction when Angular Velocity is Zero and Angular Acceleration is Negative

Nyquist Sample Criterion

Filter Length $N$ Proportional to $1/\Delta f$

Resample Ratio $M$ Controlled by $\Delta f$

$f_s > 2BW$

$f_s = 2BW + \Delta f$

Harris
Filter Length N, to Down-Sample Ratio M, N/M Ratio, Is a Constant

Filter Length at Input Sample Rate:

\[ N \approx \frac{f_s}{\Delta f} \frac{A(dB)}{22} \]

2-Sided Alias Free BW as Fraction of Output Sample Rate:

\[ 2BW = \alpha \frac{f_s}{M} \]  
(Protect from Aliasing When Down Sampling)

Aliased Transition BW, \( \Delta f \) as Fraction of Output Sample Rate:

\[ \Delta f = (1 - \alpha) \frac{f_s}{M} \]  
(Allowable Aliasing When Down Sampling)

Substitute in Filter Length at Input Sample Rate:

\[ N = \frac{f_s}{(1 - \alpha) \frac{f_s}{M}} \frac{A(dB)}{22} = \frac{1}{(1 - \alpha) / M} \frac{A(dB)}{22} \]

Dividing both sides by M

\[ \frac{N}{M} = \frac{1}{(1 - \alpha)} \frac{A(dB)}{22} \]  
\[ \frac{N}{M} \text{ (Ops/Output)} = \frac{N}{M} \text{ (Ops)} \]

Fraction of Spectrum allocated to Aliased Transition BW

N and M are Coupled and Their Ratio has Interesting Units
Sample Rate Large Compared to Bandwidth

Nyquist Rate for Filter is 40 kHz + 40 kHz = 80 kHz or fs/50
Can Perform 50-to-1 Down Sample and Still Satisfy Nyquist
Rather than Resample the output of a filter, we can resample the impulse response of the filter and the modified filter will resample the time series!
Coefficient Assignment of Polyphase Partition Mapping From 1-D to 2-D Filter

For M-to-1 resample start at Index r and Increment by M
For 3-to-1 resample start at index r and increment by 3

Extract Delays To First Non-Zero Coefficient

This mapping from 1-D to 2-D is used by Cooley-Tukey FFT. Polyphase Filters and CT-FFT are kissing cousins!
Partition of Low Pass Filter 1-Path to M-Path Transformation

\[ H(Z) = \sum_{n=0}^{N-1} h(n)Z^{-n} \]

\[ H(Z) = \sum_{r=0}^{M-1} \sum_{n=0}^{N-1} h(r + nM)Z^{-(r+nM)} \]

\[ H(Z) = \sum_{r=0}^{M-1} Z^{-r} \sum_{n=0}^{N-1} h(r + nM)Z^{-nM} \]

M-Path Partition Supports M-to-1 Down Sample
Polyphase Partition of Band Pass Filter, 1-Path to M-Path Transformation

Modulation Theorem of Z-Transform

\[ G(Z) = \sum_{n=0}^{N-1} h(n) e^{j\theta_k n} Z^{-n} = \sum_{n=0}^{N-1} h(n) (e^{-j\theta_k} Z)^{-n} = H(e^{-j\theta_k} Z) \]

\[ G(Z) = \sum_{r=0}^{M-1} \sum_{n=0}^{N-1} h(r + nM) e^{j\theta_k (r+nM)} Z^{-(r+nM)} \]

\[ G(Z) = \sum_{r=0}^{M-1} e^{j\theta_k} Z^{-r} \sum_{n=0}^{N-1} h(r + nM) e^{j\theta_k nM} Z^{-nM} \]

\[ M \cdot \theta_k = k \cdot 2\pi \]

or \[ \theta_k = k \cdot \frac{2\pi}{M} \]

\[ G(Z) = \sum_{r=0}^{M-1} e^{j\frac{2\pi}{M} k} Z^{-r} \sum_{n=0}^{N-1} h(r + nM) Z^{-nM} \]
M-Path Polyphase Band Pass Filter and M-to-1 Down Sampler
Noble Identity: Commutate M-Units of Delay followed by M-to-1 Down Sampler

M-Units of Delay at Input Rate Same as 1-Unit of Delay at Output Rate
Apply Noble Identity to Polyphase Partition

We Reduce Sample Rate M-to-1 Prior to Reducing Bandwidth
(Nyquist is Raising His Eyebrows!)

We Intentionally Alias the Spectrum. (Were you Paying Attention when they discussed the importance of anti-aliasing filters?)

M-fold Aliasing!
M-Unknowns!
M-Paths Supply M-Equations
We can the Separate Aliases!
Move Phase Spinners to Output of Polyphase Filter Paths

Want Phase Spinners as far away from Resampler as Possible
Polyphase Partition with Commutator
Replacing the “r” Delays in the “r-th” Path

Note: By moving the Phase Spinner To the Filter Output We don’t Select Desired Center Frequency Till after Down Sampling And Path Processing

This Means that The Processing for every Channel is the same till the Phase Spinner

No longer LTI, Filter now has M-Different Impulse Responses! Now LTV or PTV Filter.
Reduce Sample Rate at Input to Filter: Very Efficient Implementation!

4-MHz Input Sample Rate → 50 Path Polyphase Filter → 80-kHz Output Sample Rate

4-MHz Input
50-to-1

8 taps

\[ \phi_0 \]
\[ \phi_1 \]
\[ \phi_2 \]
\[ \phi_{48} \]
\[ \phi_{49} \]

80-kHz Output

8-Tap Filter

Select Path Weights

50-Path Coefficient Bank

400 Weights

N/M = 400/50 = 8-ops/Input

18
Down Sample to Reduce Sample Rate in Proportion to Bandwidth Reduction and Up Sample to Preserve Input Sample Rate.

4-MHZ Input Sample Rate

50 Path Polyphase Filter

80-kHZ Internal Sample Rate

50 Path Polyphase Filter

4-MHZ Input Sample Rate

8 taps

400 Tap Filter Implemented with 8 Ops/Input

4-MHZ Input Sample Rate

400 Tap Filter Implemented with 16 Ops/In-Out

4-MHZ Input Sample Rate

400 Tap Interpolator Implemented with 8 Ops/Output
4-MHz Input Sample Rate

400 Weights

\[ \Delta f = 4 \text{ kHz, } f_s = 4000 \text{ kHz} \]

H1 Filter

\[ N_1 = 4000 \text{ Taps} \]

4-MHz Output Sample Rate

\[ \phi_0 \]
\[ \phi_1 \]
\[ \phi_2 \]
\[ \phi_{48} \]
\[ \phi_{49} \]

8 taps

4-MHz

50 Path Polyphase Filter

80-kHZ Internal Rate

\[ \Delta f = 4 \text{ kHz, } f_s = 80 \text{ kHz} \]

H2 Filter

\[ N_2 = 80 \text{ Taps} \]

4-MHz Output Sample Rate

\[ \phi_0 \]
\[ \phi_1 \]
\[ \phi_2 \]
\[ \phi_{48} \]
\[ \phi_{49} \]

8 taps

4-MHz

50 Path Polyphase Filter

1-to-50

\[ \phi_0 \]
\[ \phi_1 \]
\[ \phi_2 \]
\[ \phi_{48} \]
\[ \phi_{49} \]

8 taps

4-MHz

4000 Tap Filter Implemented with 18 Ops/Input
Cascade M-to-1 and 1-to-M Polyphase Partitioned Low Pass Filter
Cascade M-to-1 and 1-to-M Polyphase Partitioned Band Pass Filter

\[ x(n,k_1) \rightarrow H_0(Z) \rightarrow e^{j0k_1\frac{2\pi}{M}} \rightarrow H_1(Z) \rightarrow e^{j1k_1\frac{2\pi}{M}} \rightarrow H_2(Z) \rightarrow e^{j2k_1\frac{2\pi}{M}} \rightarrow H_{(M-1)}(Z) \rightarrow e^{j(M-1)2k_1\frac{2\pi}{M}} \rightarrow y(nM,k_1) \rightarrow e^{j0k_1\frac{2\pi}{M}} \rightarrow H_1(Z) \rightarrow e^{j1k_1\frac{2\pi}{M}} \rightarrow H_2(Z) \rightarrow e^{j2k_1\frac{2\pi}{M}} \rightarrow H_{(M-1)}(Z) \rightarrow e^{j(M-1)2k_1\frac{2\pi}{M}} \rightarrow y(n,k_1) \]
Cascade M-to-1 and 1-to-M Polyphase Partitioned Band Pass Filter with Frequency Offset
Cascade M-to-1 and 1-to-M of Two Polyphase Partitioned Band Pass Filters
We have a method to reduce workload when the bandwidth is a small fraction of the sample rate. That won’t help us when we have a bandwidth that is large fraction of the sample rate. Possibly worse, the sample rate is extremely high.

Hmm???

The solution: We simply change the problem that we can’t solve to one we can and solve that one instead. (The Star Trek Kobayashi Scenario, Thinking Out of the Box).

We partition the bandwidth into multiple, parallel, reduced bandwidth segments so that each derives benefit from the down-sampling, up-sampling process.
Compare Traditional Implementation to Polyphase Filter Implementation

- Traditional Input Filter: 600 Taps, 600 Multiplies
- Channelizer Filter: 600 Taps
- Channelized Bandwidth
- Enabled Channels
- Disabled Channels: 8.5% of Workload
- Output Filter: 51 Multiplies, 91.5% Reduction
Phase Rotated Sum: Scaled IFFT

\[ y(nM,k) = \sum_{r=0}^{M-1} y_r(nM)e^{j\frac{2\pi}{M}rk} \]

- nM-th Time Sample From K-th Nyquist Zone
- Sum Over Row Output Time Samples
- nM-th Output Time Sample From r-th Row

Recognize Sum as Scaled IFFT of Row Vector of Time Samples at Time nM
M-Channel Analysis Channelizer: M-Narrowband Filter Spans Centered on Multiples of Output Sample Rate Aliased to Baseband with Unique Phase Profiles Separated by Phase Rotators in M-Point IFFT

Hmm... this is very good stuff....
Dual Graphs, Dual Functions
Polyphase Down Converter, (Analysis Filter Bank)
Polyphase Up Converter, (Synthesis Filter Bank)

Alias Multiple Narrow Band Signals from Different Center Frequencies to Baseband By Down Sampling

Alias Multiple Narrow Band Baseband Signals to Different Center Frequencies By Up Sampling
Channelizer Parameters

- Center frequencies, hence channel spacing, and the number of paths in filter partition defined by length $M$ of IFFT.
- Channel bandwidth and spectral characteristics, in-band ripple, out-of-band attenuation, and transition BW defined by prototype low-pass filter in polyphase partition.
- Channelizer output sample rate determined by input commutator span of $P$ inputs per $M$-point IFFT output.
- Three parameters are independent and adjustable.
Two Channelizer BW Options

Channelizer for High Quality Spectrum Analyzer

Channelizer for High Quality FDM Receiver
Overlapped Channel BW and Output Sample Rate Options

The Winner!
Filter Spectral Shape at Input Sample Rate

Filter Spectral Shape at Output Sample Rate

Maximally Decimated

Non-Maximally Decimated

Band Edge Alias

\[ f_{BW} = \frac{fs}{M} \]

\[ f_{SMPL} = fs \]

\[ f_{BW} = \frac{fs}{M} \]

\[ f_{SMPL} = \frac{fs}{M} \]

\[ f_{SMPL} = 2\left(\frac{fs}{M}\right) \]
Filter Sampled at Rate to Avoid Band Edge Aliasing

- Sample Rate = Channel Spacing
- Aliased Transition Bandwidth
- Channel Spacing
- Sample Rate = 2 Channel Spacing
Perform M/2-to-1 Resampling in M-Path Filter
Insert Circular Buffer Between Polyphase Filter and IFFT
to Align Input Data Shifting Origin with IFFT’s Origin

Circular Buffer

M Path Poly Phase Filter in $\mathbb{Z}^2$

flg=0

M-PNT IFFT

2 fs/M

fs

flg=0

$\text{flg}=0$

Circular Buffer

Phase Continuity

Perform $M/2$-to-$1$ Resampling in M-Path Filter
Insert Circular Buffer Between Polyphase Filter and IFFT
to Align Input Data Shifting Origin with IFFT’s Origin

$mT$

$0.5T$

$T$

$(m+1)T$

$0.5T$

$T$

$1.5T$

$2T$

Phase Continuity
M-Path Analysis Filter Bank with M/2 to 1 Down Sampling Non Maximally Decimated Perfect Reconstruction Filter Bank
M-Path Synthesis Filter Bank with 1 to M/2 Up Sampling
Non Maximally Decimated Perfect Reconstruction Filter Bank
Filtering in the Cascade Channelizer Domain

Looks Like Fast Convolution. **It Isn’t!**
Fast Convolution Occurs in Frequency Domain, Alters Spectrum and Returns to Time Domain. Channelizer Never Leaves Time Domain!
Impulse Response and Frequency Response
25-Enabled Ports: 5.0 MHz Bandwidth

Sample Rate
24 MHz.

IFFT Size
120 Points.

FFT Bin Width
200 kHz.

Bandwidth
Resolution
Multiples of 200 kHz.

Impulse Response

Frequency Response, ± 2.4 MHz
Impulse Response and Frequency Response

39-Enabled Ports: 7.8 MHz Bandwidth

Sample Rate 24 MHz.

IFFT Size 120 Points.

FFT Bin Width 200 kHz.

Bandwidth Resolution Multiples of 200 kHz.
Implement 600 Tap Filter with 60 Path, 600 Tap Cascade Analysis and Synthesis Filter
Filters for Next Generation *DOCSIS Standard

Signal Specifications; Complex Input Samples, Independent Upper and Lower 95 MHZ Channels

Filter Specifications

0.1 dB In-Band Ripple

80.0 dB Out-of-Band Attenuation
Complex Input Signal
Require 2-Filters for Upper Band
2800 ops/Input
Frequency Domain Filtering With Cascade M/2-to-1 Analysis and 1-to-M/2 Synthesis Channelizers

600 Tap Prototype Filter
60 Path Polyphase Partition
10 Coefficients per Path
Input Filter:
20 Operations Per Input Sample

60 Point Good-Thomas Nested Winograd FFT
165 Real Multiples
Amortized over 30 Input Samples:
5.5 Operations per Input Sample

For Input and Output Polyphase Filter and IFFT:
51 Operations per Input-Output Sample Pair

Workload: 8.5% of 600 Coefficient Tapped Delay Line Filter
With same Frequency Response!
Impulse Response, 60 Path, 600 Tap, Cascade Analysis and Synthesis Filter Bank Filter

51 Multiplies per Input Sample

Impulse Response, 600 Tap Direct Implementation Filter

600 Multiplies per Input Sample
Change Bandwidth with Binary Mask, Change Sample Rate with Transform Size

\[ \text{Reduced BW} \quad F_s^{\text{OUT}} = F_s^{\text{IN}} \]

\[ \text{Reduced BW} \quad F_s^{\text{OUT}} = (P/M) F_s^{\text{IN}} \]
Reassemble Decomposed Broadband Signals Using Short Synthesis Filters Formed by Multiple Channel Analysis Channelizer
Partitioned Spectral Components from Single Multi-Channel Analysis Filter Bank
Reassembled Wide Band Channels from Short Synthesis Channelizers

Synthesized Spectrum of 44.7 MHz Signal at $f_C = 100$ MHz

Synthesized Spectrum of 10.7 MZ Signal at $f_C = 40$ MHz

Synthesized Spectrum of 74.0 MHz Signal at $f_C = 160$ MHz

Synthesized Spectrum of 22.1 MZ Signal at $f_C = 80$ MHz

Synthesized Spectrum of 147.5 MHz Signal at $f_C = 320$ MHz
Cascade M/2-to-1 Analysis and 1-to-M/2 Synthesis Channelizers
Frequency Domain Filtering and Spectral Shuffle
Professor harris, may I be excused?  
My brain is full.
Is Open For Questions