MVDR and MPDR

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Reference Books

1. Optimum Array Processing, H. L. Van Trees
2. Stoica, P., & Moses, R. L. (2005). Spectral analysis of signals (Vol. 1). Upper Saddle River, NJ: Pearson Prentice Hall.

Narrow-band Signals

$$\mathbf{x}(\omega_c, n) = \mathbf{V}(\omega_c, \mathbf{k}_s)F_s[n] + \sum_{l=1}^{D-1} \mathbf{V}(\omega_c, \mathbf{k}_l)F_l[n] + \mathbf{Z}[n]$$

Assumptions

- $F_s[n]$, $F_l[n], l = 1, .., D-1$, and $\mathbf{Z}[n]$ are zero mean
- $E(|F_s[n]|^2) = P_s$, and $E(|F_l[n]|^2 = P_l, l = 1, \ldots, D-1$, and $E(\mathbf{Z}[n]\mathbf{Z}^{[}n]) = \sigma_z^2\mathbf{I}$
- All the signals/sources are uncorrelated with each other and over time: $E(F_l[n]F_m^*[p]) = P_l\delta[l-m]\delta[n-p]$ and $E(F_l[n]F_s^*[p]) = 0$
- The sources are uncorrelated with the noise: $E(\mathbf{Z}[n]F_l^*[m]) = \mathbf{0}$

Interference plus Noise signal Covariance

$$\mathbf{I}[n] = \sum_{l=1}^{D-1} \mathbf{V}_l F_l[n] + \mathbf{Z}[n], \quad ; \text{ where } \mathbf{V}_l = \mathbf{V}(\omega_c, \mathbf{k}_l)$$

Properties of $\mathbf{I}[n]$

- $\mathbf{I}[n]$ is zero mean
- The covariance of $\mathbf{I}[n]$ is given by

$$\mathbf{S}_n = E(\mathbf{I}[n]\mathbf{I}^H[n]) = \sum_{l=1}^{D-1} P_l \mathbf{V}_l \mathbf{V}_l^H + \sigma_z^2 \mathbf{I}$$

MVDR Beamformer

Distortionless constraint on beamformer W: $W^H \mathbf{V}_s = 1$
Implication:

$$\begin{aligned} W^H \mathbf{x}[n] &= W^H \mathbf{V}_s F_s[n] + W^H \mathbf{I}[n] && (1) \\ &= \underbrace{F_s[n]}_{\text{distortionless constraint}} + q[n], \text{ where } q[n] = W^H \mathbf{I}[n] && (2) \end{aligned}$$

Minimum Variance objective: Choose W to minimize

$$E(|q[n]|^2) = W^H \mathbf{S}_n W$$

MVDR BF design

$$\min_W W^H \mathbf{S}_n W \text{ subject to } W^H \mathbf{V}_s = 1.$$

MVDR beamformer

MVDR BF design

$$\min_{W} W^H \mathbf{S}_n W \;\text{ subject to }\; W^H \mathbf{V}_s = 1.$$

Solution: $W_{mvdr} = \frac{1}{\mathbf{V}_s^H \mathbf{S}_n^{-1} \mathbf{V}_s} \mathbf{S}_n^{-1} \mathbf{V}_s$

Derivation: Can be Obtained using Lagrange multipliers. Since we are dealing with complex weights, need to use Wirtinger calculus.

See textbook for details

Interpretation

$W_{mvdr} = \frac{1}{\mathbf{V}_s^H \mathbf{S}_n^{-1} \mathbf{V}_s} \mathbf{S}_n^{-1} \mathbf{V}_s$ tries to minimize
$E(|q[n]|^2) = E(|W^H \mathbf{I}[n]|^2) = W^H \mathbf{S}_n W$, where $\mathbf{S}_n = \sum_{l=1}^{D-1} P_l \mathbf{V}_l \mathbf{V}_l^H + \sigma_z^2 \mathbf{I}$

It will try to place nulls at angular locations consistent with the interference plane waves if σ_z^2 is small.

If the number of antennas is greater than or equal to D, i.e. $N \geq D$, the MVDR BF can null out all the $(D-1)$ interferers.

If $N < D$, the MVDR BF will attempt to control the depth of the nulls to minimize interference.

If σ_z^2 is large compared to the power in the interfering plane waves, then $\mathbf{S}_n \approx \sigma_z^2 \mathbf{I}$ and hence $W_{mvdr} \propto \mathbf{V}_s$

MVDR by SINR maximization

$$W_{mvdr} = \arg\max_W \frac{|W^H \mathbf{V}_s|^2 P_s}{W^H \mathbf{S}_n W} = \arg\max_W \frac{|W^H \mathbf{V}_s|^2}{W^H \mathbf{S}_n W}$$

Note that the solution is not unique but is unique to scale. The scale can be chosen to satisfy the distortionless constraint.
Since $\mathbf{S}_n$ is positive definite it admits a factorization $\mathbf{S}_n = LL^H$, where L is a $N \times N$ matrix, a square root of $\mathbf{S}_n$
The square root is not unique but invertible because $\mathbf{S}_n$ is positive definite. Then

$$W^H \mathbf{S}_n W = W^H L L^H W = \tilde{W}^H \tilde{W}, \text{ where } \tilde{W} = L^H W \text{ or } W = L^{-H} \tilde{W}$$

The optimization over W can be replaced by an optimization over $\tilde{W}$

$$\tilde{W}_o = \arg\max_{\tilde{W}} \frac{|\tilde{W}^H L^{-1} \mathbf{V}_s|^2}{\tilde{W}^H \tilde{W}}$$

By the Cauchy-Schwarz inequality, $\tilde{W}_o \propto L^{-1}\mathbf{V}_s$ or $\tilde{W}_o = \beta L^{-1}\mathbf{V}_s$
Then $W_{mvdr} = L^{-H}\tilde{W}_o = \beta L^{-H} L^{-1} \mathbf{V}_s = \beta \mathbf{S}_n^{-1} \mathbf{V}_s$
The distortional less constraint $W_{mvdr}^H \mathbf{V}_s = 1$ leads to $\beta = \frac{1}{\mathbf{V}_s^H \mathbf{S}_n^{-1} \mathbf{V}_s}$
Hence $W_{mvdr} = \frac{1}{\mathbf{V}_s^H \mathbf{S}_n^{-1} \mathbf{V}_s} \mathbf{S}_n^{-1} \mathbf{V}_s$

Challenges with MVDR

$$W_{mvdr} = \frac{1}{\mathbf{V}_s^H \mathbf{S}_n^{-1} \mathbf{V}_s} \mathbf{S}_n^{-1} \mathbf{V}_s$$

The main challenge is estimating $\mathbf{S}_n$?

This requires coordination and may not always be possible.

This leads to MPDR, minimum power distortionless response beamformer.

Most books refer to MPDR as MVDR.

MPDR, minimum power distortionless response beamformer

MPDR very similar to MVDR with respect to the constraint.

Distortionless constraint on beamformer W: $W^H \mathbf{V}_s = 1$
Implication:

$$W^H \mathbf{x}[n] = W^H \mathbf{V}_s F_s[n] + W^H \mathbf{I}[n] \tag{3}$$

$$= \underbrace{F_s[n]}_{\text{distortionless constraint}} + q[n], \text{ where } q[n] = W^H \mathbf{I}[n] \tag{4}$$

Minimum Power objective: Choose W to minimize $E(|W^H|\mathbf{x}[n]|^2)$, the power at the output of the beamformer

$$E(|W^H \mathbf{x}[n]|^2) = W^H \mathbf{S}_x W, \text{ where } \mathbf{S}_x = P_s \mathbf{V}_s \mathbf{V}_s^H + \mathbf{S}_n$$

MPDR BF design

$$\min_W W^H \mathbf{S}_x W \text{ subject to } W^H \mathbf{V}_s = 1.$$

MPDR beamformer

MPDR BF design

$$\min_{W} W^H \mathbf{S}_x W \;\text{ subject to }\; W^H \mathbf{V}_s = 1.$$

Solution: $W_{mpdr} = \frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s} \mathbf{S}_x^{-1} \mathbf{V}_s$

Derivation: Same as MVDR with $\mathbf{S}_x$ replacing $\mathbf{S}_n$

Benefit:

- $\mathbf{S}_x$ is easier to determine making it computationally attractive

$$\mathbf{S}_x \approx \frac{1}{L} \sum_{n=1}^{L-1} \mathbf{x}[n]\mathbf{x}^H[n]$$

- Same $\mathbf{S}_x$ is needed if you change your mind on direction of interest. Can deal with multiple signals of interest with considerable ease.

Relationship between MPDR and MVDR

For uncorrelated sources $W_{mpdr} = W_{mvdr}$

Proof is based on the Matrix Inversion Lemma

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

Note that

$$\mathbf{S}_x = P_s\mathbf{V}_s\mathbf{V}_s^H + \mathbf{S}_n = \mathbf{S}_n + \mathbf{V}_s P_s \mathbf{V}_s^H$$

MPDR needs inverse of $\mathbf{S}_x$. Using the matrix inversion lemma with $A = \mathbf{S}_n$, $B = \mathbf{V}_s$, $C = P_s$, and $D = \mathbf{V}_s^H$, we have

$$\mathbf{S}_x^{-1} = \mathbf{S}_n^{-1} - \mathbf{S}_n^{-1}\mathbf{V}_s(\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s + \frac{1}{P_s})^{-1}\mathbf{V}_s^H\mathbf{S}_n^{-1}$$

$W_{mpdr} = W_{mvdr}$

$$\mathbf{S}_x^{-1} = \mathbf{S}_n^{-1} - \mathbf{S}_n^{-1}\mathbf{V}_s(\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s + \frac{1}{P_s})^{-1}\mathbf{V}_s^H\mathbf{S}_n^{-1}$$

$$\begin{aligned}
\mathbf{S}_x^{-1}\mathbf{V}_s &= \mathbf{S}_n^{-1}\mathbf{V}_s - \mathbf{S}_n^{-1}\mathbf{V}_s(\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s + \frac{1}{P_s})^{-1}\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s \\
&= \mathbf{S}_n^{-1}\mathbf{V}_s - \mathbf{S}_n^{-1}\mathbf{V}_s\frac{\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s}{\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s + \frac{1}{P_s}} \\
&= \frac{\frac{1}{P_s}}{\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s + \frac{1}{P_s}}\mathbf{S}_n^{-1}\mathbf{V}_s = \beta\mathbf{S}_n^{-1}\mathbf{V}_s \text{ where } \beta = \frac{\frac{1}{P_s}}{\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s + \frac{1}{P_s}}
\end{aligned}$$

Note that

$$\mathbf{V}_s^H\mathbf{S}_x^{-1}\mathbf{V}_s = \beta\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s$$

Hence

$$W_{mpdr} = \frac{1}{\mathbf{V}_s^H\mathbf{S}_x^{-1}\mathbf{V}_s}\mathbf{S}_x^{-1}\mathbf{V}_s = \frac{1}{\beta\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s}\beta\mathbf{S}_n^{-1}\mathbf{V}_s = \frac{1}{\mathbf{V}_s^H\mathbf{S}_n^{-1}\mathbf{V}_s}\mathbf{S}_n^{-1}\mathbf{V}_s = W_{mvdr}$$

Spatial Power Spectrum using MPDR

- Beamsteering and measuring power at the output of BF, i.e. $E(|(W_d \odot \mathbf{V}(\psi_T))^H \mathbf{x}[n]|^2)$. FFT based processing for ULA
- MPDR based spatial power spectrum estimation: Measure power at the output of the MPDR BF given by $W_{mpdr} = \frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s} \mathbf{S}_x^{-1} \mathbf{V}_s$

$$
\begin{aligned}
E(|W_{mpdr}^H \mathbf{x}[n]|^2) &= W_{mpdr}^H \mathbf{S}_x W_{mpdr} \\
&= \left(\frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s} \mathbf{V}_s^H \mathbf{S}_x^{-1} \right) \mathbf{S}_x \left(\mathbf{S}_x^{-1} \mathbf{V}_s \frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s} \right) \\
&= \frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s} \left(\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{S}_x \mathbf{S}_x^{-1} \mathbf{V}_s \right) \frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s} \\
&= \frac{1}{\mathbf{V}_s^H \mathbf{S}_x^{-1} \mathbf{V}_s}
\end{aligned}
$$