Optimal Scheduling Policies and the Performance of the CDF Scheduling

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***Abstract*—We seek to characterize the performance of scheduling policies for wireless systems that are based on Cumulative Density Functions (CDF) of the channels. We first derive optimal scheduling policies for a two-user system under max-sum rate and max-min rate performance criteria subject to the same temporal resource constraints as those under CDF-based policies. The behaviors of the CDF schemes are then compared against those of the optimal policies. We illustrate the differences in scheduling decision boundaries as well as the sub-optimality in rate performance of the CDF-based policies.**

***Index Terms*—CDF, Scheduling, Fairness**

I. Introduction

DYNAMIC user scheduling has always been a very difficult task in wireless systems. A good scheduling policy must be able to take advantage of multiuser diversity to achieve high system throughput and at the same time guarantee service fairness. These tasks become much more challenging in the next generation wireless network as the channels experienced by the users will have very different characteristics due to the multi-tier, heterogeneous nature of the systems. The CDF-based scheduling policy introduced in [1] in which the users are selected based on how good their current channels are relative to *their own* channel conditions, independent of the specific channel distributions emerges to be a good choice for such diverse systems.

Despite its salient properties, CDF scheduling characteristics are not very well understood comparing to other popular scheduling schemes such as the *Proportional Fairness* scheme, which has been studied extensively in the literature. Beside some basic properties introduced in the original paper [1], several additional performance features examined in [2], and scaling laws in [3], many questions regarding CDF scheduling characteristics remain unanswered. Better understanding of the properties of the CDF scheme is crucial in bringing this technique to the practical implementation in next generation wireless systems. In this paper, we will further examine CDF scheduling characteristics by deriving the optimal scheduling policies with respect to common metrics and comparing and contrasting CDF scheduling performance against the optimality. For fair comparisons, the optimal policies are also subjected to the same temporal resource constraints achieved by CDF scheduling. For mathematical tractability, we only derive the solutions for a two-user system. Given the diverse and dynamic nature of the wireless systems, the optimal policies can become extremely complex to analyze and/or implement in reality for large systems. As a results, there have been few works in the literature discussing optimal scheduling policies. Most practical existing scheduling methods are sub-optimal, ad-hoc in nature. The work in [4] does consider scheduling optimality. However, the authors in this work consider only the max sum-rate criterion and they do not derive explicit closed functional form for the scheduling decision boundary, which we need for comparing against CDF scheduling. In [4], the decision boundary is learned via stochastic approximations.

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II. Background and Scheduling Criteria

We start with a quick review of the CDF scheduling scheme. Consider a set of K users sharing a common channel resource. Let X_k be the SNR for the user k. Perform the transformation $U_k = F_{X_k}(X_k)$, where $F_{X_k}(x)$ is the CDF of X_k. The CDF scheduling policy selects a user to serve according to: $k^* = \underset{k}{\operatorname{argmax}}\, U_k^{1/w_k}$, where w_k is the time fraction allocated to user k. It is known that this policy performs well. It is, nonetheless, a sub-optimal scheme [2] with respect to rate performance due to the fact that it is not formulated for rate optimization.

Let us now consider the problem of maximizing the average sum rate of a K-user system subject to the same user probability of access constraints $0 \le w_k \le 1$ with $\sum_{k=1}^K w_k = 1$. Let $\mathbf{X} = [X_1, \dots, X_K]^T$, and $\mathbf{x} = [x_1, \dots, x_K]^T$, the vector of SNR realizations. The users' instantaneous rates R_k and the average sum rate R_s for the system are then

$$R_k = \begin{cases} \log(1 + X_k), & \text{when user k selected} \\ 0, & \text{otherwise} \end{cases}$$

$$R_s = \mathbf{E}\left[\sum_{k=1}^K R_k(X_k)\right] = \int_{0 \preceq \mathbf{x} \prec \infty} \sum_{k=1}^K R_k(X_k) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint pdf of $\mathbf{X}$. The max sum rate problem can be formulated as follows:

$$\mathcal{V}^* = \underset{\mathcal{V} \in \mathcal{P}_x}{\operatorname{argmax}} \sum_{k=1}^K \int_{\mathcal{V}_k} R_k(x_k) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$
$$\text{s.t.} \int_{\mathcal{V}_k} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = w_k$$

where $\mathcal{V}_k$ is the region in the positive orthant $0 \le x_1, \dots, x_K \le \infty$ corresponding to user k being selected, $\mathcal{V} = \{\mathcal{V}_k\}_{k=1}^K$, the set of all $\mathcal{V}_k$'s, and $\mathcal{P}_x$ are the set of all partitions of the positive orthant. Let $u_k = F_{X_k}(x_k)$, where $F_{X_k}(x)$ is the cumulative distribution function (CDF) of the SNR of user k. Perform a change of variables from $x_1, \dots, x_K$

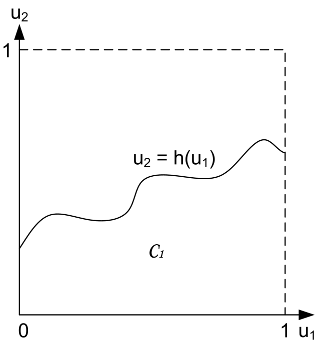

Fig. 1: Scheduling Decision Boundary

to $u_1, \ldots, u_K$, and let $\mathcal{C}_k$ be the region in the u_1-u_K space (a.k.a. the *u*-space) where user k is selected, $\mathcal{C} = \{\mathcal{C}_k\}_{k=1}^K$, and $\mathcal{P}_u$ the set of all partitions of the hypercube $0 \leq u_k \leq 1$. With the users independently distributed, this selection scheme can be re-formulated as follows in the u-space:

$$\mathcal{C}^* = \underset{\mathcal{C}\in\mathcal{P}_u}{\text{argmax}} \sum_{k=1}^{K} \int_{\mathcal{C}_k} \log[1 + F_{X_k}^{-1}(u_k)] d\mathbf{u} \quad \text{s.t. Volume}(\mathcal{C}_k) = w_k \tag{1}$$

Next, we consider the max min-rate criterion. Under this criterion, it is necessary to compare user achievable rates. Yet, even if the user channels are i.i.d, users with different temporal resource constraints receive different average rates. Consequently, it is necessary to define a metric suitably normalized by the resource allocation to facilitate the required user performance comparison.

Definition 1. The *allocation-normalized rate,* $\bar{R}_k$ of user k with an allocation constraint $w_k : \sum_{k=1}^K w_k = 1$ is defined as

$$\bar{R}_k = \frac{R_k}{K w_k}$$

where K is the number of users in the system, R_k is the average rate achieved by user k. The max-min selection policy can now be defined as:

$$\mathcal{V}^* = \underset{\mathcal{V}\in\mathcal{P}_x}{\text{argmax}} \min_k \bar{R}_k \quad \text{s.t.} \int_{\mathcal{V}_k} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = w_k$$

This problem can be reformulated in the u-space as follows:

$$\mathcal{C}^* = \underset{\mathcal{C}\in\mathcal{P}_u}{\text{argmax}} \min \frac{1}{w_k} \int_{\mathcal{C}_k} \log[1 + F_{X_k}^{-1}(u_k)] d\mathbf{u} \quad \text{s.t. Volume}(\mathcal{C}_k) = w_k \tag{2}$$

III. Max Sum-Rate Scheduling for a Two-User System

In this section, we derive the optimal scheduling scheme for a two-user system under the system sum-rate criterion. With the number of users $K = 2$, the objective function of (1) becomes

$$R_s = \int_{\mathcal{C}_1} \log[1 + F_{X_1}^{-1}(u_1)] d\mathbf{u} + \int_{\mathcal{C}_2} \log[1 + F_{X_2}^{-1}(u_2)] d\mathbf{u}$$

$$= \int_{\mathcal{C}_1} \left(\log[1 + F_{X_1}^{-1}(u_1)] - \log[1 + F_{X_2}^{-1}(u_2)]\right) d\mathbf{u} + \int_{\mathcal{C}_1\cup\mathcal{C}_2} \log[1 + F_{X_2}^{-1}(u_2)] d\mathbf{u}$$

The selection scheme that maximizes the system sum rate can be formulated as follows

$$\mathcal{C}_1^* = \underset{\mathcal{C}_1}{\text{argmax}} \int_{\mathcal{C}_1} \Big(\log[1 + F_{X_1}^{-1}(u_1)] - \log[1 + F_{X_2}^{-1}(u_2)]\Big) du_1 du_2 \quad \text{s.t. Area}(\mathcal{C}_1) = w_1 \tag{3}$$

Now let $\mathcal{C}_1$ be a region bounded by $0 \leq u_1 \leq 1, 0 \leq u_2 \leq h(u_1)$, where $0 \leq h(u_1) \leq 1, \forall u_1 \in [0,1]$ as in figure 1. The solution to problem (3) is stated in theorem 1.

Theorem 1. *The optimal decision boundary for the sum-rate criterion has the following form:*

$$h^*(u_1) = F_{X_2}\left(\lambda'[1 + F_{X_1}^{-1}(u_1)] - 1\right) \tag{4}$$

$$\textit{s.t.} \int_0^1 h^*(u_1) du_1 = w_1 \tag{5}$$

Proof: see Appendix A. ■

The condition (5) can be used to solve for λ' in order to satisfy the access probability constraint.

IV. Max Min-Rate Scheduling for a Two-User System

For a two-user system under the max min-rate criterion, problem (2) becomes the following

$$\mathcal{C}_1^* = \underset{\mathcal{C}_1}{\text{argmax}} \min \left\{ \frac{1}{w_1} \int_{\mathcal{C}_1} \log[1 + F_{X_1}^{-1}(u_1)] du_1 du_2 \quad A - \frac{1}{w_2} \int_{\mathcal{C}_1} \log[1 + F_{X_2}^{-1}(u_2)] du_1 du_2 \right\} \quad \text{s.t. Area}(\mathcal{C}_1) = w_1 \tag{6}$$

where $A \triangleq \frac{1}{w_2} \int_{\mathcal{C}_1\cup\mathcal{C}_2} \log[1 + F_{X_2}^{-1}(u_2)] du_1 du_2$. With the region $\mathbf{C}_1$ defined in section III, problem (6) becomes

$$h^*(u_1) = \underset{h(u_1),t}{\text{argmax}}\, t$$

$$\text{s.t.: } \frac{1}{w_1} \int_0^1 h(u_1) \log[1 + F_{X_1}^{-1}(u_1)] du_1 \geq t$$

$$A - \frac{1}{w_2} \int_0^1 du_1 \underbrace{\int_0^{h(u_1)} \log[1 + F_{X_2}^{-1}(u_2)] du_2}_{I[h(u_1)]} \geq t \tag{7}$$

$$\int_0^1 h(u_1) du_1 = w_1$$

The solution to problem (7) is summarized in theorem 2.

Theorem 2. *The optimal decision boundary for the max min-rate criterion has one of the following three forms*

1) *Equal rates are not possible, user 1 is too strong.*

$$u_2 = h^*(u_1) = w_1$$

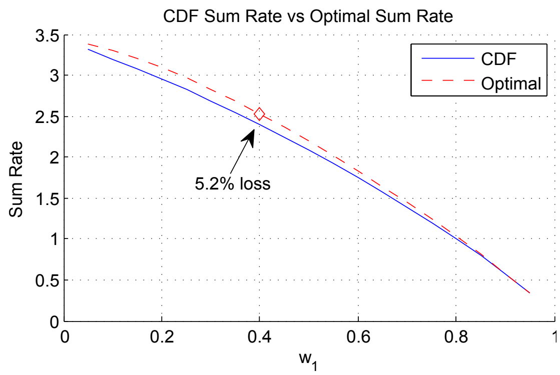

Fig. 2: Non-IID Users: CDF Scheduling's Sum Rate Loss Under High User Channel Discrepancy

2) *Equal rates are not possible, user 2 is too strong.*

$$u_1 = g^*(u_2) = w_2 = 1 - w_1$$

3) *Equal rates are possible.*

$$h^*(u_1) = F_{X_2}\left(\lambda_3[(1 + F_{X_1}^{-1}(u_1))]^{\frac{\lambda_1 w_2}{\lambda_2 w_1}} - 1\right), \quad (8)$$

where λ_1, λ_2, *and* λ_3 *can be found from the following system of equations*

$$\begin{cases} \lambda_1 + \lambda_2 = 1, 0 \leq \lambda_1, \lambda_2 \leq 1 \\ \int_0^1 h^*(u_1) du_1 = w_1 \\ \bar{R}_1(h^*) = \bar{R}_2(h^*) \end{cases} \quad (9)$$

Proof: see Appendix B. ■

V. CDF Scheduling Performance Comparisons

It is well known that CDF scheduling is optimal when the user channels are *independent and identically distributed* (iid) and all users have the same resource allocations. Thus, in order to characterize its sub-optimality, we consider different allocation constraints under both non-iid and iid user channels. Assuming Rayleigh fading for both users, we have

$$f_{X_1}(x) = \frac{1}{C_1} e^{-\frac{x}{C_1}},\ F_{X_1}(x) = 1 - e^{-\frac{x}{C_1}}$$
$$f_{X_2}(x) = \frac{1}{C_2} e^{-\frac{x}{C_2}},\ F_{X_2}(x) = 1 - e^{-\frac{x}{C_2}}$$
$$F_{X_1}^{-1}(x) = -C_1 \log(1-x),\ F_{X_2}^{-1}(x) = -C_2 \log(1-x)$$

where C_1, C_2 are the mean SNR values.

First, let us consider the max sum-rate criterion. Result (4) becomes

$$h^*(u_1) = 1 - e^{-[\lambda(1 - C_1 \log(1-u_1)) - 1]/C_2} \quad (10)$$

Plugging (10) into the constraint (5) and after some algebraic manipulation, we get

$$1 - \frac{C_2 e^{\frac{1-\lambda}{C_2}}}{C_2 + \lambda C_1} = w_1 \quad (11)$$

Equation (11) can then be solved numerically for λ. For the *non-iid* case where there is a large difference between the two users average SNR's, the sum rate for CDF policy versus the optimal selection is shown in Figure 2. In this case

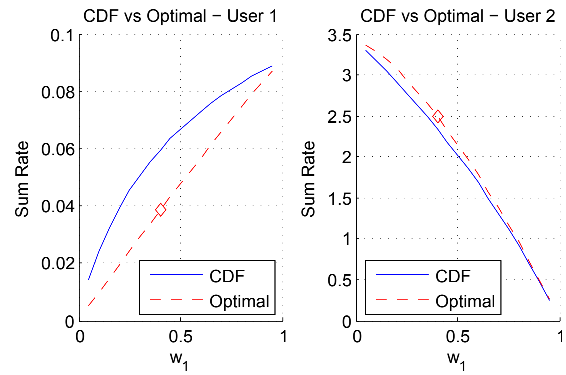

Fig. 3: Non-IID Users: CDF Scheduling Allocates More Rate To Weak Users

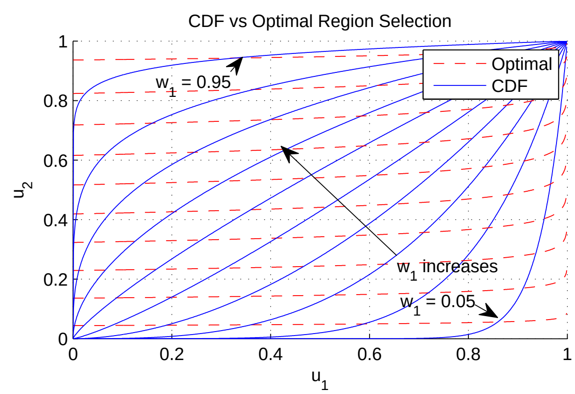

Fig. 4: Non-IID Users: Decision Boundaries for CDF Scheduling vs the Optimal Policy

(very high user discrepancy with $C_1 = 0.1$, $C_2 = 50$), the maximum rate loss for CDF is around 5%. Figure 3 shows that the CDF scheme allocates more rate to the weak user (user 1), while sacrificing the strong user's performance. Figure 4 illustrates the differences in scheduling decision boundaries for CDF scheduling and the optimal policy for different resource allocations. For the *iid* case ($C_1 = C_2 = 50$ in this example), the CDF policy is optimal when the allocation is $w_1 = w_2 = 0.5$. Otherwise, it has a small loss compared to the optimal allocation ($\approx 1.2\%$ in this case) as seen in Figure 5.

Next, we consider the max min-rate criterion under the same iid channel setting. The normalized rate for CDF scheme vs.

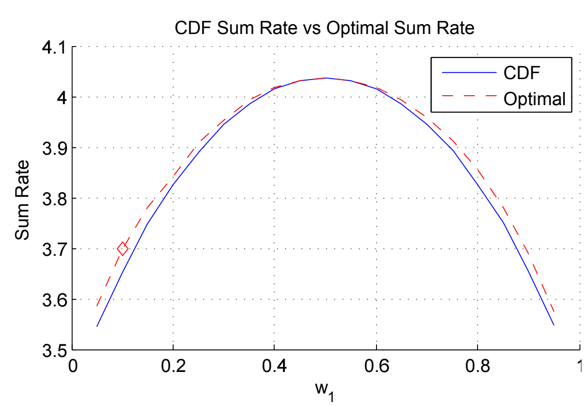

Fig. 5: Max Sum-Rate Criterion: CDF Scheduling's Sum Rate Loss Under IID Channels

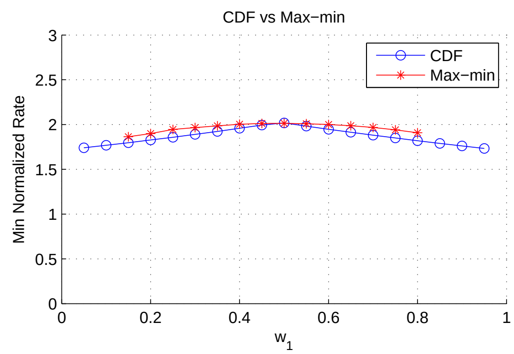

Fig. 6: Max Min-Rate Criterion: CDF Scheduling's Min Rate Loss Under IID Channels

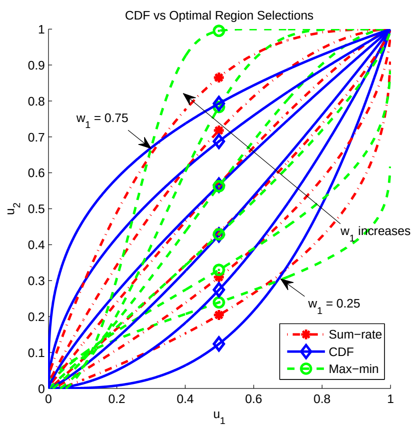

Fig. 7: Decision Boundaries for CDF vs Optimal Policies Under IID Channels

the optimal max-min policy is shown on figure 6. The CDF loss is around 5% in this case. The comparison of allocation boundaries between CDF and the optimal policies for both max sum-rate and max min-rate criteria for different allocation constraints is shown on Figure 7. It can be seen that even in the i.i.d case, the CDF policy is only *close* to the optimal policies when user allocation constraints w_k are close to each other, i.e. $w_k \approx 1/K$.

VI. Conclusion

In this paper we derive the optimal decision boundaries for a two-user system under both max sum-rate and max min-rate criteria. The performance of the CDF scheduling policy is compared against both optimal policies. It can be seen while the CDF scheme is not optimal in general, the loss in performance is small, which may be a good tradeoff for its low complexity. The study of the optimal scheduling boundaries and CDF scheduling performance for a system with more than two users is part of our future work.

Appendix A
Proof of Theorem 1

For brevity, we provide only a sketch of the proof. Problem (3) can be rewritten as follows:

$$h^*(u_1) = \underset{h(u_1)}{\text{argmax}} \int_0^1 du_1 \Bigg(h(u_1)\log[1+F_{X_1}^{-1}(u_1)] - \underbrace{\int_0^{h(u_1)} \log[1+F_{X_2}^{-1}(u_2)]du_2}_{I[h(u_1)]} \Bigg)$$

$$\text{s.t.} \int_0^1 h(u_1)du_1 = w_1$$

Form the following Lagrangian:

$$\mathcal{L}(h,\lambda) \triangleq \int_0^1 \Bigg(h(u_1)\log[1+F_{X_1}^{-1}(u_1)] - I[h(u_1)] + \lambda h(u_1) \Bigg) du_1 - \lambda w_1$$

Following the principle of *Calculus of Variations*, we let $h(u_1) = h^*(u_1) + \epsilon\delta(u_1)$ and set $\frac{\partial}{\partial\epsilon}\mathcal{L}(\epsilon,\lambda)\Big|_{\epsilon=0} = 0, \forall\delta(u_1)$. This leads to the following result after some algebraic steps:

$$\log[\lambda'(1+F_{X_1}^{-1}(u_1))] - \log[1+F_{X_2}^{-1}(h^*(u_1))] = 0$$

where $\log\lambda' \triangleq \lambda$. This results in the optimal boundary in (4).

Appendix B
Proof of Theorem 2

From (7), we can form the following Lagrangian:

$$\begin{aligned}
\mathcal{L}(h,\lambda_1,\lambda_2,\lambda_3,t) &= t + \\
&+ \lambda_1\left(\frac{1}{w_1}\int_0^1 h(u_1)\log[1+F_{X_1}^{-1}(u_1)]du_1 - t\right) \\
&+ \lambda_2\left(A - \frac{1}{w_2}\int_0^1 du_1 \underbrace{\int_0^{h(u_1)} \log[1+F_{X_2}^{-1}(u_2)]du_2}_{I[h(u_1)]} - t\right) \\
&+ \lambda_3\left(\int_0^1 h(u_1)du_1 - w_1\right)
\end{aligned}$$

where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. Again, following the principle of *Calculus of Variations*, we let $h(u_1) = h^*(u_1) + \epsilon\delta(u_1)$ and obtain the results in theorem 2 by setting $\frac{\partial}{\partial\epsilon}\mathcal{L}(\epsilon,\lambda_1,\lambda_2,\lambda_3,t)\Big|_{\epsilon=0} = 0, \forall\delta(u_1)$.

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