FRAME DESIGN USING FOCUSS WITH METHOD OF OPTIMAL DIRECTIONS (MOD)

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ABSTRACT

The equation $\mathbf{b} = \mathbf{Ax} + \mathbf{n}$ where the columns of $\mathbf{A}$ form an overcomplete set, i.e. the system is underdetermined, and with a sparsity constraint on $\mathbf{x}$ can be important to solve in many applications. It can be used as a convenient signal representation model useful for compression, and it can also be a model for the true underlying system that produced the available dataset $\mathbf{b}$. It is hard enough to solve the equation for a sparse solution when $\mathbf{A}$ is known. An even harder problem is to try to find both the $\mathbf{A}$ and the $\mathbf{x}$ that produced the data set $\mathbf{b}$, which is the only available data. This paper shows that a frame design algorithm, Method of Optimal Directions (MOD), proposed by Engan et al. [1], used with a noise robust version of FOCUSS we proposed in [2] works well for reconstructing the true $\mathbf{A}$ from the dataset $\mathbf{b}$. The MOD algorithm has already produced good results on designing frames for compression of ElectroCardioGram (ECG) signals [3, 4], and the results in this paper provides complimentary evidence of its good properties.

1. INTRODUCTION

In this paper we consider the signal model:

$$\mathbf{b} = \mathbf{Ax} + \mathbf{n}, \tag{1}$$

where $\mathbf{b}$ is a $N \times 1$ data vector, $\mathbf{A}$ is an $N \times M$ matrix where $M > N$, $\mathbf{x}$ is an $M \times 1$ coefficient vector and $\mathbf{n}$ is an $N \times 1$ noise vector. The columns of the matrix $\mathbf{A}$ form an overcomplete set, and spans the space R^N, so the equation (1) is underdetermined. The columns of $\mathbf{A}$ are not a basis but a frame [5].

Equation (1) shows up in several important applications. In lossy signal compression, $\mathbf{n}$ represents the reconstruction error, and $\mathbf{Ax} = \tilde{\mathbf{b}}$ the approximation of the signal vector $\mathbf{b}$. Making a representation of the signal like this is equivalent to doing transform based coding if $\mathbf{A}$ is $N \times N$ and invertible. It is the same as gain-shape Vector Quantization (VQ) if $\mathbf{A}$ is $N \times M$ where $M >> N$ and exactly one entry in $\mathbf{x}$ is nonzero, and VQ if the nonzero coefficient is equal to 1. In our case it is frame based coding, where $\mathbf{A}$ is $N \times M$, $M > N$ and $\mathbf{x}$ is sparse but has more than one nonzero entry. In the case of lossy signal compression the quality of the approximation, for a given sparsity of $\mathbf{x}$ and a specified $\mathbf{A}$, is of primary importance.

On the other hand, if we want to find the true underlying structure that produced the data, finding the correct $\mathbf{A}$ is essential. Olshausen and Field have done work where they try to find a model of some of the response properties of neurons in primary visual cortex [6]. If the theory is that the neurons actually work according to the model of Equation (1) with a sparse $\mathbf{x}$, it is desirable to find the true $\mathbf{A}$ and $\mathbf{x}$. Gorodinitsky and Rao [7] have done work using this sparsity model for functional imaging of the brain using EEG or MEG signals. Other applications could be signal reconstruction and denoising, or blind source separation with fewer sensors than sources. In the latter case some work is done by Lewicki and Sejnowski [8].

If the frame, $\mathbf{A}$, is known it is still a hard problem to find a sparse $\mathbf{x}$. With a sparsity constraint, $\mathbf{x}$ can be limited to have only r nonzero elements, where $r << M$. Finding the optimal entries is an NP complete problem, and this requires extensive calculation [9]. A suboptimal technique is preferable in order to limit the computational complexity. There exist several different vector selection methods dealing with this problem. They can be divided into sequential (greedy) and parallel vector selection methods [10]. Earlier experiments with training of frames done by Engan et al.[3, 4] using the Method of Optimal Directions (MOD) was done in association with a greedy algorithm for vector selection called Orthog-

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onal Matching Pursuit OMP [11]. In the experiments in this paper, we use a regularized FOCUSS in noise as the vector selection algorithm, which is a parallel method. In general parallel methods appear to perform better, but are more computational complex than sequential methods.

2. REGULARIZED FOCUSS IN NOISE

Solving the noise free inverse problem:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{2}$$

when the equation system is underdetermined, i.e. $\dim(\mathbf{A}) = N \times M$ where $N \leq M$, and with a sparsity constraint on $\mathbf{x}$, led to the developing of a method called FOCUSS, for **FOC**al **U**nderdetermined **S**ystem **S**olver [7] by using $\ell_{(p\leq 1)}$ diversity measure given by [12]

$$E^{(p)}(\mathbf{x}) = \operatorname{sgn}(p)\sum_{i=1}^{n}|\mathbf{x}[i]|^p, \quad p \leq 1. \tag{3}$$

The original FOCUSS gives an exact solution to the problem. In some application, like using the model for compression purposes, it can be desirable to allow for an error to achieve a solution that is very sparse, i.e. $\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{n}$. The data can also be polluted with noise, so that an error has to be accepted if a good reconstruction is wanted.

A variation of the FOCUSS algorithm that allows noise has been discussed in [13]. This is an iterative algorithm:

$$\begin{aligned}
\mathbf{W}_{k+1} &= \operatorname{diag}(|\mathbf{x}_k[i]|^{1-\frac{p}{2}}) && (4)\\
\mathbf{q}_{k+1} &= \arg\min_{\mathbf{q}} \|\mathbf{A}\mathbf{W}_{k+1}\mathbf{q} - \mathbf{b}\|^2 + \lambda\|\mathbf{q}\|^2 && (5)\\
\mathbf{x}_{k+1} &= \mathbf{W}_{k+1}\mathbf{q}_{k+1} && (6)
\end{aligned}$$

where Equation 5 is a regularization optimization problem. The two terms in Equation (5) are a function of the parameter λ, and the regularization problem is a compromise between sparsity and error in the representation.

In [2] we proposed a modification of the original FOCUSS that allows a noise vector like in Equation 1. The algorithm use regularization to find a trade off between minimizing the error $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|$, and maximizing the sparsity of $\mathbf{x}$. The regularization was done as a combination of the discrepancy principle and the L-curve principle [14]. With a criteria on the quality of fit, we get the discrepancy principle:

$$\min_{\mathbf{x}} E^{(p)}(\mathbf{x}) \ \text{ subject to } \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\| \leq \epsilon.$$

The L-curve was introduced by Hansen in [15] as a method for finding the parameter λ in the regularization problem:

$$\min_{\mathbf{x}}\{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \lambda\|\mathbf{x}\|^2\}, \tag{7}$$

and this can easily be translated to the regularization problem of Equation 5 which we want to solve. The theory of the L-curve poses that a plot of $\|\mathbf{q}\|^2$ versus $\|\mathbf{A}\mathbf{W}_{k+1}\mathbf{q} - \mathbf{b}\|^2$ for different λ will have an L shape and that a good λ is the one corresponding to the corner in the L. Further more it is suggested [15, 14, 16] that the corner of the L-shaped curve can be found by finding the maximum curvature.

With some knowledge of the noise, or alternatively the target SNR, the regularized FOCUSS in noise combined these two criteria to develop a robust method that is shown to work well in [2].

3. METHOD OF OPTIMAL DIRECTIONS (MOD)

The MOD is an iterative training algorithm inspired by the Generalized Lloyd Algorithm (GLA) [17]. MOD was first presented in [1] in a somewhat different context. MOD was originally motivated for designing frames to be used for compression purposes. In the MOD, the frame design problem is tackled by dividing each iteration in the training algorithm into two, similar to what's done in the GLA. This can be summarized as:

1. $\mathbf{A}$ and $\mathbf{b}_l$, $l = 1, 2 \ldots K$ are known. Find $\mathbf{x}_l$, $l = 1, 2 \ldots K$ by using a vector selection algorithm.

2. $\mathbf{x}_l$ and $\mathbf{b}_l$, $l = 1, 2 \ldots K$ are known. Find the best possible $\mathbf{A}$.

Finding the best possible frame when $\mathbf{x}_l$ and $\mathbf{b}_l$, $l = 1, 2 \ldots K$ are known is a much more complicated task than using the centroid conditions as done in the GLA. The method is summarized below, but more detail is found in [4]. At each iteration the frame is updated as

$$\mathbf{A}_i = \mathbf{A}_{i-1} + \tilde{\mathbf{R}}_{rx} \cdot \tilde{\mathbf{R}}_{xx}^{-1}, \tag{8}$$

where $\tilde{\mathbf{R}}_{rx}$ is the estimate of the cross correlation between the residual and the coefficients, and $\tilde{\mathbf{R}}_{xx}$ a estimation of the autocorrelation matrix for the coefficients. Let $\mathbf{R} = [\mathbf{r}_1\ \mathbf{r}_2 \ldots \mathbf{r}_K], \mathbf{r}_l = \mathbf{b}_l - \tilde{\mathbf{b}}_l$ be the residual matrix, and $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_K]$ be the coefficient matrix. Then:

$$\begin{aligned}
\tilde{\mathbf{R}}_{rx} &= \mathbf{R} \cdot \mathbf{X}^T && (9)\\
\tilde{\mathbf{R}}_{xx} &= \mathbf{X} \cdot \mathbf{X}^T && (10)
\end{aligned}$$

Each frame vector $\mathbf{a}_j$ is adjusted in the optimal way with respect to reduction of the total MSE, and this is why the method is called MOD, method of optimal directions.

The main steps in the algorithm are as follows:

1. $\mathbf{b}_l, l = 1, 2 \ldots K$ is the training set. Begin with an initial frame $\mathbf{A}_0$ of size $N \times M$. Assign counter variable $i = 1$.

2. Approximate each training vector, $\mathbf{b}_l$, using a sparse vector selection algorithm:

$$\tilde{\mathbf{b}}_l = \mathbf{A}\mathbf{x} = \sum_{j=1}^{M} x_l(j)\mathbf{a}_j, \tag{11}$$

   where $x_l(j)$ is the coefficient corresponding to vector $\mathbf{a}_j$. $\mathbf{x}$ is a sparse vector, that is many of the $x_l(j)$'s are zero.

   Find the residuals.

3. Given the approximations and residuals, find a new frame $\mathbf{A}_i$.

4. Find the new approximations, and calculate the new residuals. If (stop-criterion = FALSE) $\Rightarrow$ $i = i + 1$, go to step 3. Otherwise stop.

Several stop-criteria can be used; for example maximum number of iterations or almost constant MSE. The convergence properties are not yet fully understood, but in all the experiments in this paper the algorithm converges.

4. EXPERIMENTS AND RESULTS

Frame reconstruction experiments are done with a data set, $\mathbf{b}_l$, $l = 1, 2 \ldots 1000$, where the data vectors are made from the following equation:

$$\mathbf{A}_{org}\mathbf{x}_l = \mathbf{b}'_l \tag{12}$$

$$\mathbf{b}_l = \mathbf{b}'_l + \mathbf{n}_l \tag{13}$$

The experiments are done using an 20×30 original matrix, $\mathbf{A}_{orig}$ with random entries, chosen from a normal distribution with mean zero and variance one. The columns in $\mathbf{A}_{orig}$ are normalized to one. The set, $\mathbf{b}'_l$, $l = 1, 2 \ldots 1000$ are made as a linear combination of r randomly picked vectors from $\mathbf{A}_{orig}$. This means that the set $\mathbf{x}_l$, $l = 1, 2 \ldots 1000$, consists of vectors with r nonzero elements, coefficients. The nonzero coefficients are Gaussian random variables with zero mean and unit variance.

Experiments are done with r fixed at 4 and then at 7. Experiments with r varying within the training vector set is also done. In this case r is uniformly distributed between 1 and 10, this gives a mean $\bar{r} = 5.5$. The $\mathbf{b}'_l$'s are normalized. The data vectors, $\mathbf{b}_l$, are $\mathbf{b}'_l + \mathbf{n}_l$ where $\mathbf{n}_l$ is a noise vector with Gaussian random entries with zero mean and different variances deciding the Signal to Noise Ratio (SNR) in the experiment. Experiments were done without noise and with SNR at 20 dB.

In the reconstruction experiment the only available data is the training set $\mathbf{b}_l, l = 1, 2 \ldots 1000$. An initial frame is constructed by using a normalized version of the first 30 vectors from the training set: $\mathbf{A}_0$. The frame that the training converges to is called $\mathbf{A}_{conv}$. If $\mathbf{A}_{conv} \simeq \mathbf{A}_{orig}$ the procedure has worked well in reconstructing the generative model of the underdetermined system with sparsity constraint.

The training of the frames is done by using MOD on the training set, and by using the regularized FOCUSS in noise as the vector selection algorithm required in the MOD. Two factors are of main interest, the number of vectors used in the approximation, i.e. the sparsity factor, and the error. Therefore the average number of vectors, and the Mean Squared Error (MSE) is plotted as a function of training iterations in the experiments. In all the experiments, the training converges completely.

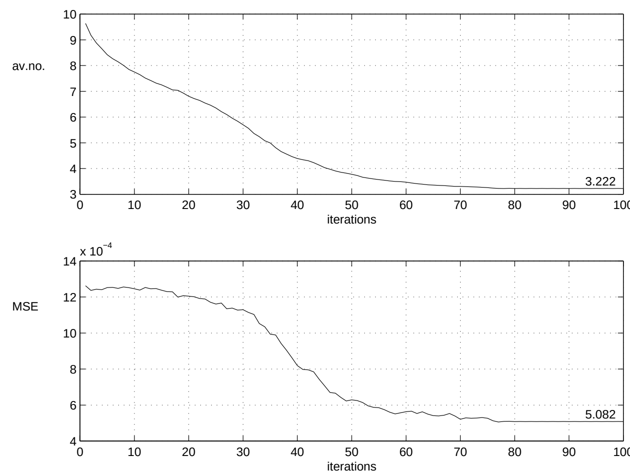

Figure 1: $r = 4$, no noise.

Figures 1 and 2 show plots of the sparsity factor and the MSE as a function of training iterations for the experiments without noise for $r = 4$ and $r = 7$ respectively. In the experiment with $r = 4$ all the 30 frame vectors where reconstructed from the data, so that $\mathbf{A}_{conv} = \mathbf{A}_{orig}$[1].

In the experiment with $r = 7$, 29 of the 30 frame vectors were reconstructed to within 1% error. In both these experiments, it can be seen from the figure that the average number of vectors used in the approximation of a signal vector at convergence is lower than the number of vectors used to *produce* the data set. When $r = 4$ the sparsity factor converges at 3.222 instead of 4, and for $r = 7$ it converges at 5.103 instead of 7.

[1] a small difference, (1%) measured by the norm of the error for each vector is allowed in all the experiments

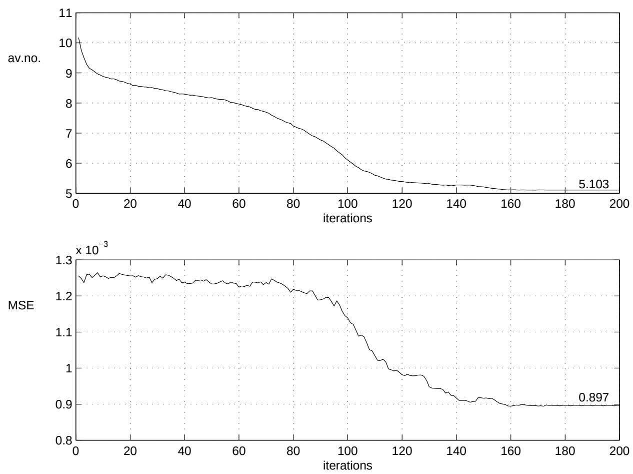

Figure 2: $r = 7$, no noise.

This means that the reconstruction of the set $\mathbf{x}_l, l = 1, 2 \ldots 1000$ is not quite accurate. This is not surprising since we use a version of FOCUSS that allows for noise. Even if we have not added noise in the training set here, we start with the wrong $\mathbf{A}$, and have to allow for noise if we want sparse solutions. In terms of compression this is a good result since the reconstructed coefficient vector is even *sparser* than the true coefficient vector, and this is true for all the experiments presented here. For model reconstruction this may not be wanted, but since the true $\mathbf{A}_{orig}$ is reconstructed, a FOCUSS version that allows less noise can be used in combine with the true $\mathbf{A}_{orig}$, or the $\mathbf{A}_{conv}$ after the training. This may result in a more accurate reconstruction of the set $\mathbf{x}_l,\ l = 1, 2 \ldots 1000$.

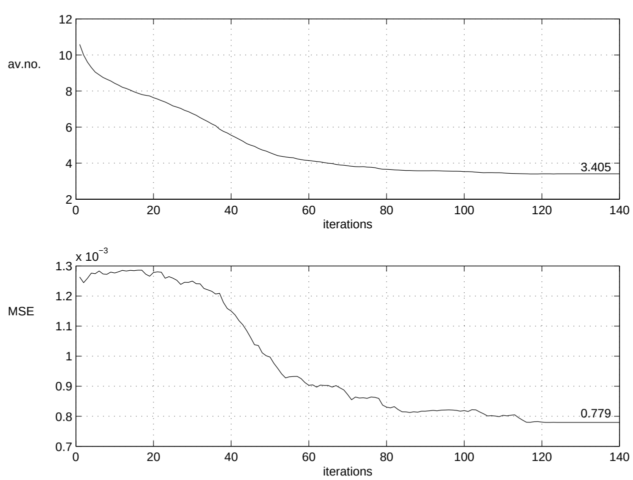

Figure 3: $r = 4$, noise level 20 dB.

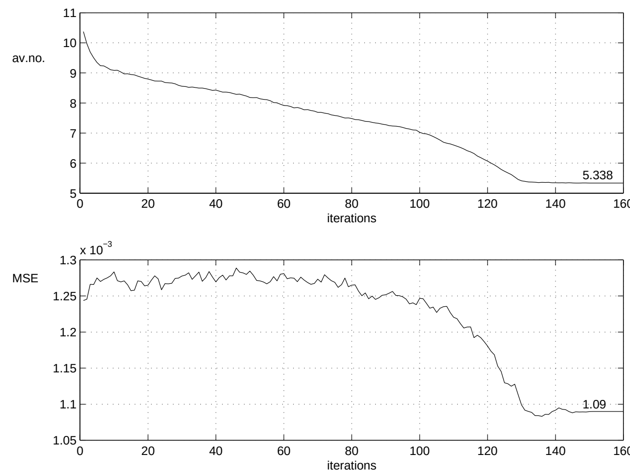

Figure 4: $r = 7$, noise level 20 dB.

The MSE in all the experiments is also plotted, and converges to a lower value than the start value. Since we use the regularized FOCUSS in noise as the vector selection algorithm, we allow for noise in our attempt to model the data vector. The regularized FOCUSS require a target SNR as input, and the result will have an SNR somewhere around this value. Therefore it is not expected for the MSE values to drop dramatically, as it does in the training experiments in [1, 4]. In the training experiments in [1, 4] the OMP is used as vector selection algorithm, and a sparsity criteria is used. This way the sparsity factor of the solution is constant, and the MSE drops. In the experiments in this paper both the sparsity factor and the MSE drops during training. The most dramatic development is in the sparsity factor due to the target SNR when using the regularized FOCUSS, but also the MSE drops when we approach convergence.

Figure 3 shows the experiments with $r = 4$ and noise level at 20 dB. In this experiment all the 30 frame vectors where found, so that $\mathbf{A}_{conv} = \mathbf{A}_{orig}$. The experiment showed in Figure 4 was done with $r = 7$ and noise level at 20 dB, and here 29 of the 30 frame vectors from $\mathbf{A}_{orig}$ was reconstructed in $\mathbf{A}_{conv}$. The sparsity factor in these two experiments converges at a bit higher values than in the two earlier experiments and this can be due to the noise that is added to the training set.

For the two experiments with uniformly distributed r, showed in Figure 5 and 6, all the 30 frame vectors were reconstructed, so that $\mathbf{A}_{conv} = \mathbf{A}_{orig}$. Also here the sparsity factor at convergence, 3.752 and 3.969, is less than the $\bar{r} = 5.5$, and it is less in the no noise case than in the 20 dB case.

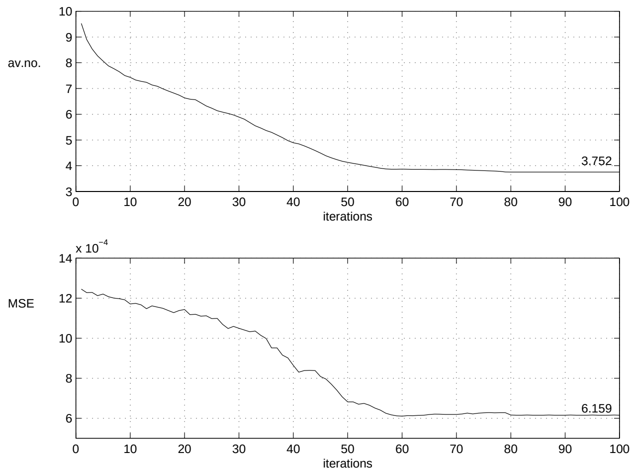

Figure 5: Uniformly distributed r, no noise.

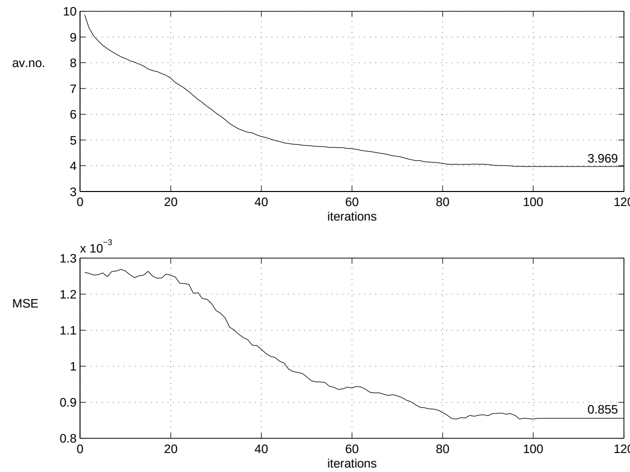

Figure 6: Uniformly distributed r, noise level 20 dB.

5. CONCLUSION

We have proposed a frame learning algorithm that is a combination of MOD and regularized FOCUSS. This method can be used to reconstruct the $\mathbf{A}$ that was the original generative frame which together with a set of sparse $\mathbf{x}$'s was used to produce the data set we have in hand. We have called this model reconstruction of underdetermined systems with sparsity constraints. This can be useful when we know that a physical system has this true underlying sparse structure, and we only can access the data vectors. It is also a good indication that MOD works very well with a good vector selection algorithm. The MOD algorithm has already produced good results on designing frames for compression of ElectroCardioGram (ECG) signals [3, 4], and the results in this paper provides complimentary evidence of its good properties.

6. REFERENCES

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