ECE 275A Homework # 3, Due Thursday 10/29/2009

Reading:
In addition to the lecture material presented in class, students are to read and study the following:

A. The material in Section 4.11 of Moon & Stirling Sections 8.1–8.9 of Kay. Pay particular attention to the material in Kay Section 8.7 on Recursive (aka Sequential) Least Squares (RLS) and Kay Section 8.6 on Order-Recursive Least Squares (ORLS).

B. The Gram-Schmidt orthogonalization process and its relationship to the QR factorization described in Sections 2.15 and 5.3.1–5.3.4 of Moon & Stirling.

C. The Singular Value Decomposition (SVD), described in Sections 7.1–7.5 of Moon & Stirling. Note that the notation used in lecture corresponds to the notation used on page 371 of Moon & Stirling by taking $S = \Sigma_1$ and $0 = \Sigma_2$. A good concise source of information on the SVD can also be found in the paper


which can be downloaded from IEEE Xplore, which is accessible for free (along with very many other useful databases such as INSPEC, jstor, and MathSciNet) at http://libraries.ucsd.edu/sage/databases.html

if you have UCSD network privileges (and proxy server permission if you work from home). You should understand the proof of the derivation of the SVD as you could be asked on an exam to reproduce it.

D. The Total Least Squares (TLS) algorithm described in Section 7.7 of Moon & Stirling.

Homework:

1. Let $A$ be a linear operator $A : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are complex finite-dimensional Hilbert spaces with $\dim(\mathcal{X}) = n$ and $\dim(\mathcal{Y}) = m$.

   (a) Prove that $Ax = 0$ if and only if $\langle y, Ax \rangle = 0, \forall y$.

   Note that a variety of equivalent statements follow from this, such as $My = 0$ if and only if $\langle x, My \rangle = 0, \forall x$, which can be further particularized by taking $M = A^*$.

   (b) Prove that $A$ is onto iff $A^*$ is 1-1 and that $A$ is 1-1 iff $A^*$ is onto. You can use the fact derived in class that $r(A) = r(A^*)$. 

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(c) Prove that $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$.

(d) Prove that $\mathcal{N}(A) = \mathcal{N}(A^*A)$. Use this fact to prove that $A^*A$ is an invertible operator if and only if $A$ is one-to-one and that $AA^*$ is invertible if and only if $A$ is onto.

(e) Prove that a (general nonorthogonal) projection operator $P : \mathcal{X} \to \mathcal{X}$ is unique and is idempotent.

(f) Prove that an orthogonal projection operator, $P : \mathcal{X} \to \mathcal{X}$, is self-adjoint, $P = P^*$, where $P^*$ is the adjoint operator of $P$.

(g) Prove that if $n = m$ and $A$ is full rank, then $A^+ = A^{-1}$.

(h) Let $A : \mathcal{X} \to \mathcal{Y}$ be represented as an $m \times n$ matrix where $\mathcal{X}$ has a hermitian positive definite inner product weighting matrix $\Omega$ and $\mathcal{Y}$ has an inner product weighting matrix $W$. (a) Prove that when $r(A) = m$, $A^+$ is independent of $W$. (b) Prove that when $r(A) = n$, $A^+$ is independent of $\Omega$.

(i) (a) Prove that any two particular least-squares solutions $x_p$ and $x_p'$ to the linear inverse problem $Ax = y$ must orthogonally project to the same minimum norm least-squares solution,

$$P_{\mathcal{R}(A^*)}x_p = P_{\mathcal{R}(A^*)}x_p'.$$

(This justifies the claim made in class that the minimum-norm least-squares solution is uniquely determined from the orthogonal projection of any particular least-squares solution onto $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$.)

(b) Prove that the set of all (particular) least-squares solutions is equal to the linear-variety (translated subspace),

$$\hat{x} + \mathcal{N}(A) \triangleq \{\hat{x}\} + \mathcal{N}(A)$$

2. Let $\mathcal{X} = \mathbb{C}^{n \times n}$ be the Hilbert space of $n \times n$ complex matrices, $X$, with Frobenius inner product

$$\langle X_1, X_2 \rangle = \text{tr} X_1^H X_2.$$

Let $\mathcal{V}$ be the set of symmetric (not hermitian) $n \times n$ complex matrices $V = V^T$, $V \in \mathcal{X} = \mathbb{C}^{n \times n}$.

Finally, define the mapping

$$P(\cdot) : \mathcal{X} \to \mathcal{X} \text{ by } P(X) \triangleq \frac{X + X^T}{2}.$$

(a) Prove that $\mathcal{V}$ is a (Hilbert) subspace of $\mathcal{X}$ and give its dimension. Prove that the set of hermitian matrices is not a subspace.
(b) Prove that $P(\cdot)$ is the orthogonal projection of $\mathcal{X}$ onto the subspace $\mathcal{V}$. Assume at the outset that you know nothing about $P(\cdot)$ other than its definition.\(^1\)

(c) Determine the orthogonal projector of $\mathcal{X}$ onto $\mathcal{V}^\perp$ and categorize the elements of the subspace $\mathcal{V}^\perp$.

3. Let $\mathcal{X} = \text{Sym}(\mathbb{C}, n) \subset \mathbb{C}^{n \times n}$ be the vector space of symmetric $n \times n$ complex matrices\(^2\) with Frobenious inner product

$$\langle X_1, X_2 \rangle = \text{tr} X_1^H X_2.$$  

Let $\mathcal{Y} = \mathbb{C}^{n \times m}$ be the Hilbert space of $n \times m$ complex matrices with inner product

$$\langle Y_1, Y_2 \rangle = \text{tr} Y_1^H Y_2.$$  

Finally for a given full row-rank $n \times m$ matrix $A$, definite the mapping

$$\mathcal{A}(\cdot) : \mathcal{X} \rightarrow \mathcal{Y} \text{ by } \mathcal{A}(X) \triangleq XA, \text{ rank}(A) = n.$$  

(a) Prove that the least-squares solution to the inverse problem

$$Y = \mathcal{A}(X)$$

is necessarily unique and is the solution to the the (matrix) Lyapunov equation

$$XM + M^T X = \Lambda$$  \hspace{1cm} (1)

with

$$M \triangleq AA^H \text{ and } \Lambda \triangleq Y A^H + (YA^H)^T.$$  \hspace{1cm} (2)

(b) The Matlab Controls Toolbox provides a numerical solver for the Lyapunov Equation (1). Mathematically, it can be readily shown that a unique solution to the Lyapunov equation is theoretically given by

$$X = \int_0^\infty e^{-Mt} \Lambda e^{-Mt} dt$$  \hspace{1cm} (3)

provided that the real parts of the eigenvalues of $M$ are all strictly greater than zero.\(^3\)

\(^1\)Note that as a projection operation the domain and codomain of $P$ are both $\mathcal{X}$. For the claim to be true, what must be the range of $P$?

\(^2\)In the previous problem, the set of symmetric, complex matrices was proven to be a Hilbert subspace. Here, we treat this set as a Hilbert space in its own right.

\(^3\)This is a sufficient condition to ensure that the integral on the right hand side of equation (3) exists and is finite.
i. For a square matrix $A$, heuristically show from the Taylor series expansion of $e^{At}$ that $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$ and that $(e^{At})^T = e^{AT} t$.

ii. Justify the claim that the unique solution to (1) and (2) is given by (3).

Comment: Most of Problems 2 and 3 were given on a previous ECE275A midterm.

4. Prove the following theorem:

**Theorem A.** Let the nullspace of $B$ and the range of $C$ be disjoint. Then $C = 0$ iff $BC = 0$.

5. Assume that all Hilbert spaces are finite dimensional and complex. Recall the four Moore-Penrose Pseudoinverse necessary and sufficient conditions for a linear operator $M$ to be the pseudoinverse of an linear operator $A$.

   (a) Prove that $(A^*)^+ = (A^+)^*$.  
   (b) Suppose that $A$ is rank-deficient. Prove that $A^+ = A^*(AA^*)^+$ and $A^+ = (A^*A)^+ A^*$. (Theorem A to be proved above can be useful here.)
   (c) Let $\alpha$ be a possibly zero-valued scalar (real or complex). Determine its pseudoinverse. (Prove your result.)
   (d) Let $D$ be a diagonal (possibly complex) matrix, $D = \text{diag}(d_1 \cdots d_n)$. Determine the pseudoinverse of $D$, assuming that $D$ maps between unweighted inner product spaces (why is this last condition important?).
   (e) Let $\Sigma$ be a (possibly complex) square block matrix of the form,

   $\Sigma = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$.  \hspace{1cm} (4)

   Assume that $S$ is also square. Determine the pseudoinverse of $\Sigma$, assuming that $\Sigma$ maps between unweighted inner product spaces.
   (f) Let $A$ and $C$ have the very special property that they are unitary, $A^{-1} = A^*$ and $C^{-1} = C^*$. (When $A$ and $C$ map between real unweighted Hilbert spaces, we say that they are orthogonal). Show that in this case we have that

   $$(ABC)^+ = C^* B^+ A^*.$$ 

   Note that this is not true for general, arbitrary matrices $A$, $B$, and $C$. Also note that generally,

   $$(ABC)^+ \neq C^+ B^+ A^+,$$

   except, of course, for special cases, such as the unitary case presented above or when the matrices are all invertible.

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(g) Let the complex \( m \times n \) matrix \( A \) be a mapping between unweighted complex Hilbert spaces with SVD \( A = U \Sigma V^H \), where \( \Sigma \) is of the form (4) with the \( r \times r \) diagonal matrix \( S \) containing the \( r \) nonzero singular values, \( \sigma_i \), of \( A \) as its diagonal elements. Prove that
\[
A^+ = \frac{1}{\sigma_1} v_1 u_1^H + \cdots + \frac{1}{\sigma_r} v_r u_r^H.
\]

6. For each of the following matrices, a) determine the SVD using geometric thinking and pencil and paper only; b) give orthogonal bases and projection operators for each of the four fundamental subspaces; c) give the pseudoinverse. Assume that the standard inner product, \( \langle x, y \rangle = x^T y \), holds on all spaces.

\[
A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 3 & 0 \end{pmatrix}.
\]


8. **Directed reading assignment.** Based on your reading of Sections Moon 4.11 and Kay 8.7, describe (do not derive), in words and mathematically, the Recursive (aka Sequential) Least Squares (RLS) algorithm, clearly defining all your terms.

9. **Directed reading assignment.** Based on your reading of Section 8.6 of Kay, describe (do not derive), in words and mathematically, the Order-Recursive Least Squares (ORLS) algorithm, clearly defining all your terms.

10. **Directed reading assignment.** Based on your reading of Section 7.7 of Moon, describe (do not derive) in words what the Total Least Squares (TLS) algorithm is, clearly stating all the mathematical steps involved in obtaining a TLS solution.

**Comment:**
The RLS, ORLS, and TLS algorithms which you are to describe in problems 8–10 are usually discussed at greater length in the course ECE251BN, Adaptive Filtering, when it is taught by Professor Bhaskar Rao. I highly recommend this course.