Meyer 4.1.1. Is the subset of $\mathbb{R}^n$ a vector subspace? Since $\mathbb{R}^n$ is a vector space, from the proof on page 162 of Meyer, we only have to prove closure under addition (A1) and closure under multiplication (M1). To prove these properties, we consider two arbitrary vector $x = [x_1 \cdots x_n]^T$ and $y = [y_1 \cdots y_n]^T$ in $\mathbb{R}^n$ subject to the indicated constraints, and arbitrary real scalar $\alpha$.

(a) $\{x \mid x_i \geq 0\}$. This set does not satisfy M1, since if we multiply by $\alpha = -1$ we go out of the set.

(b) $\{x \mid x_1 = 0\}$. This set satisfies A1 and M1 since if $x_1$ and $y_1$ are 0, then $x_1 + y_1 = 0$, and $\alpha x_1 = 0$. So it is a vectors subspace.

(c) $\{x \mid x_1x_2 = 0\}$. This set does not satisfy A1, since $(x_1 + y_1)(x_2 + y_2)$ is not necessarily 0 even if $x_1x_2$ and $y_1y_2$ are both 0.

(d) $\{x \mid \sum_{j=1}^{n} x_j = 0\}$. This set satisfies A1 and M1 since if $\sum x_i = 0$ then $\sum x_i + y_i = \sum x_i + \sum y_i = 0$, and $\sum \alpha x_i = \alpha \sum x_i = 0$. So it is a vector subspace.

(e) $\{x \mid \sum_{j=1}^{n} x_j = 1\}$. This set obviously does not satisfy either A1 or M1.

(f) $\{x \mid Ax = b, A \neq 0, b \neq 0\}$. This is not a subspace because subspaces have to contain the zero vector.

Meyer 4.1.2. Is the subset of $\mathbb{R}^{n \times n}$ a vector subspace? Addition is ordinary matrix addition, and scalar multiplication is ordinary (element-wise) multiplication by a real number. We consider arbitrary matrices $A$ and $B$ with elements $a_{ij}$ and $b_{ij}$, and scalar $\alpha$.

(a) Symmetric matrices. This satisfies A1 and M1 since if $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, then $a_{ij} + b_{ij} = a_{ji} + b_{ji}$, and $\alpha a_{ij} = \alpha a_{ji}$.

(b) Diagonal matrices. This satisfies A1 and M1 since addition of diagonal matrices and multiplication by a constant preserve diagonality.

(c) Nonsingular matrices. This does not satisfy A1 since if $A$ is nonsingular, then $-A$ is nonsingular, but $A + (-A) = 0$ (the zero matrix), which is obviously singular.
(d) Singular matrices. This does not satisfy A1 since for
\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]
(both singular), we have \( A + B = I \), which is nonsingular.

(e) Triangular matrices. This does not satisfy A1: Take \( A \) nontrivially upper-triangular and \( B \) nontrivially lower-triangular. Then \( A + B \) has both upper- and lower-triangular elements.

(f) Upper-triangular matrices. Similar to the case of diagonal matrices, this does satisfy A1 and M1.

(g) Matrices that commute with a given matrix \( A \). Yes. Take \( B \) and \( C \), satisfying \( AB = BA \) and \( AC = CA \). Then because matrices are linear operators, it is easily shown that A1 and M1 hold.

(h) Matrices satisfying \( A^2 = A \). No. Consider \( \alpha A \) for \( \alpha = \frac{1}{2} \) and \( A \neq 0 \), and show that M1 does not hold.

(i) Matrices satisfying \( \text{trace}(A) = 0 \). Yes. This set satisfies A1 and M1 because trace is a linear operator.

**Meyer 4.1.8.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be subspaces of \( \mathcal{V} \).

(a) Show that \( \mathcal{X} \cap \mathcal{Y} \) is a subspace. Again we only need to show conditions A1 and M1. If \( x \) and \( y \) are arbitrary elements in \( \mathcal{X} \cap \mathcal{Y} \), then \( x \in \mathcal{X} \) and \( y \in \mathcal{X} \), so \( x + y \in \mathcal{X} \). Also, \( x \in \mathcal{Y} \) and \( y \in \mathcal{Y} \), so \( x + y \in \mathcal{Y} \). So \( x + y \) is in both \( \mathcal{X} \) and \( \mathcal{Y} \), i.e. \( x + y \in \mathcal{X} \cap \mathcal{Y} \). So A1 is satisfied. Also, \( \alpha x \) is in both \( \mathcal{X} \) and \( \mathcal{Y} \) if \( x \) is, so M1 is satisfied.

(b) Show that \( \mathcal{Y} \cup \mathcal{Y} \) is not necessarily a subspace. Take for example the subsets of \( \mathbb{R}^2 \), \( \mathcal{X} = \{(x_1, x_2) \mid x_1 = 0\} \) and \( \mathcal{Y} = \{(x_1, x_2) \mid x_2 = 0\} \). These are both subspaces, but \( \mathcal{X} \cup \mathcal{Y} \) (the set of all points lying strictly on the coordinate axes) is not a subspace.

The proof that if \( \mathcal{X}, \mathcal{Y} \) are vector subspaces of \( \mathcal{V} \), then \( \mathcal{X} + \mathcal{Y} \) is a vector subspace of \( \mathcal{V} \) is given on pages 166-167 of Meyer.